

DISPERSIVE MASS TRANSPORT IN OSCILLATORY
AND UNIDIRECTIONAL FLOWS

by

ROBERT BRUCE TAYLOR III

A DISSERTATION PRESENTED TO THE GRADUATE
COUNCIL OF THE UNIVERSITY OF FLORIDA IN PARTIAL
FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

1974

TO JUNE

ACKNOWLEDGMENTS

The author would like to thank Dr. Robert G. Dean, Professor of the Department of Civil and Coastal Engineering, who supervised the research. Dr. Dean's many hours of encouragement and patient teaching will not be forgotten. Appreciation is also extended to Dr. Wayne C. Huber, Associate Professor of the Department of Environmental Engineering Sciences, whose interest and suggestions were of a great help to the author.

Ms. Pat Hult and Mrs. Wanda Smith worked many arduous hours on short notice to type a highly professional final draft of the manuscript. Special thanks are given to Mrs. Evelyn Hill, who assisted in many ways including the typing of the rough draft of the manuscript, and to Ms. Denise Frank, who did the drafting.

This research was sponsored by the Florida Power and Light Company through a grant to the Coastal and Oceanographic Engineering Laboratory, University of Florida. Both the funds made available by the Sponsor and the use of the COE Laboratory facilities are greatly appreciated. All computer work associated with this project was done on the Northeast Regional Data Center's IBM System/370-165.

TABLE OF CONTENTS

	Page
Acknowledgments.	iii
List of Tables	vi
List of Figures.	vii
Key to Symbols and Abbreviations	ix
Abstract	xv
CHAPTER	
I INTRODUCTION	1
I-A General Background.	1
I-B Discussion of Previous Work on Dispersive Mass Transport.	3
I-B.1 Unidirectional Flow.	3
I-B.2 Oscillatory Flow	12
I-C Description and Scope of the Present Work	21
II CONSIDERATIONS FOR APPLICATION OF TAYLOR'S DISPERSION ANALYSIS TO FREE SURFACE FLOWS.	22
II-A General.	22
II-B Unidirectional Flow.	27
II-C Oscillatory Flow	30
II-D Applications of Dispersion Coefficients Predicted by Taylor's Method	33
III DISPERSIVE MASS TRANSPORT IN AN INFINITELY WIDE RECTANGULAR CHANNEL	34
III-A Steady Unidirectional Flow.	34
III-A.1 Velocity Distribution.	34
III-A.2 Concentration Distribution	37
III-A.3 Dispersive Mass Transport.	38
III-B Oscillatory Flow.	41
III-B.1 Velocity Distribution.	41
III-B.2 Concentration Distribution	45
III-B.3 Dispersive Mass Transport.	47
III-C Discussion of Two Dimensional Shear Flow Results. . .	51
III-C.1 Dispersion in Unidirectional Flow as a Limit of the Oscillatory Flow Case . . .	51
III-C.2 Characteristics of Dispersive Mass Transport in Oscillatory Flow.	56

CHAPTER	Page
IV DISPERSIVE MASS TRANSPORT IN RECTANGULAR CHANNELS OF FINITE WIDTH.	66
IV-A Oscillatory Flow.	66
IV-A.1 Velocity Distribution	67
IV-A.2 Concentration Distribution.	71
IV-A.3 Dispersive Mass Transport	74
IV-B Steady Unidirectional Flow.	90
IV-B.1 Velocity Distribution	90
IV-B.2 Concentration Distribution.	93
IV-B.3 Dispersive Mass Transport	95
V COMPARISON OF RESULTS WITH FIELD AND LABORATORY DATA	100
V-A Unidirectional Flow.	100
V-B Oscillatory Flow	106
VI SUMMARY AND CONCLUSIONS	116
APPENDIX	
A DERIVATION OF NORMALIZING EXPRESSIONS FOR TWO DIMENSIONAL OSCILLATORY SHEAR FLOW SOLUTION	120
A-1 Surface Amplitude of Velocity, $u_{\max}$	120
A-2 Surface Excursion Length, L.	124
B MATHEMATICAL DETAILS FOR THREE DIMENSIONAL OSCILLATORY SHEAR FLOW DISPERSION	125
B-1 Velocity Solution.	125
B-2 Concentration Solution	128
B-3 Dispersive Mass Transport.	134
C DERIVATION OF $u_{\max}$ NORMALIZING EXPRESSION FOR THREE DIMENSIONAL SHEAR FLOW SOLUTION	137
REFERENCES.	140
BIOGRAPHICAL SKETCH	143

LIST OF TABLES

Table		Page
1	PREDICTED RESONANT PEAK DATA FOR LONGITUDINAL DISPERSION COEFFICIENT IN THREE DIMENSIONAL OSCILLATORY SHEAR FLOW.	83
2	LIMITING WIDTH-TO-DEPTH RATIOS FOR APPLICATION OF THE TWO DIMENSIONAL OSCILLATORY SHEAR FLOW DISPERSION COEFFICIENT.	87
3	OBSERVED DISPERSION DATA AND PREDICTED $\langle E_x \rangle_T$ VALUES USING (III-59)	106
4	OBSERVED DISPERSION DATA AND PREDICTED $\langle E_x \rangle_T$ VALUES FOR THE MERSEY NARROWS	110
5	OBSERVED DISPERSION DATA AS REPORTED BY SEGALL AND CORRESPONDING PREDICTED $\langle E_x \rangle_T$ VALUES . . .	113

LIST OF FIGURES

Figure		Page
1	DEFINITION SKETCH.	22
2	DEFINITION SKETCH FOR TWO DIMENSIONAL SHEAR FLOW	34
3	NON-DIMENSIONAL DISPERSION COEFFICIENT $\langle E_x \rangle_T / E_L$, AS A FUNCTION OF T'_L	53
4	DISPERSION IN UNIDIRECTIONAL FLOW AS A LIMIT OF THE OSCILLATORY FLOW CASE	55
5	COMPARISON OF DISPERSION FOR DIURNAL AND SEMI-DIURNAL PERIODS OF OSCILLATION WITH SAME u_{max}	58
6	COMPARISON OF DISPERSION FOR DIURNAL AND SEMI-DIURNAL PERIODS OF OSCILLATION WITH SAME SURFACE EXCURSION LENGTH	60
7	NON-DIMENSIONAL LONGITUDINAL DISPERSION COEFFICIENT E'_x , AS A FUNCTION OF T'_z FOR TWO DIMENSIONAL OSCILLATORY SHEAR FLOW	61
8	NON-DIMENSIONAL VELOCITY AND CONCENTRATION VARIATION COMPONENT PROFILES FOR $T'_z = 0.5$	63
9	NON-DIMENSIONAL VELOCITY AND CONCENTRATION VARIATION COMPONENT PROFILES FOR $T'_z = 1.58$	64
10	NON-DIMENSIONAL VELOCITY AND CONCENTRATION VARIATION COMPONENT PROFILES FOR $T'_z = 5.0$	65
11	DEFINITION SKETCH FOR THREE DIMENSIONAL SHEAR FLOW . .	67
12	NON-DIMENSIONAL LONGITUDINAL DISPERSION COEFFICIENT E'_x , AS A FUNCTION OF T'_z FOR FULL RANGE OF T'_c VALUES ...	79
13	NON-DIMENSIONAL LONGITUDINAL DISPERSION COEFFICIENT E'_x , AS A FUNCTION OF T'_z FOR $T'_c = 1.0, 0.67, 0.33$ AND 0.1	81
14	NON-DIMENSIONAL LONGITUDINAL DISPERSION COEFFICIENT E'_x AS A FUNCTION OF T'_z FOR $T'_c = 0.1, 0.01, 0.001,$ 0.0001 , and $\leq 1 \times 10^{-6}$	82

LIST OF FIGURES (Continued)

Figure		Page
15	NON-DIMENSIONAL VERTICAL MIXING TIME, T'_Z , AT WHICH MAXIMUM E'_X OCCURS AS A FUNCTION OF T'_C	84
16	PEAK E'_X AS A FUNCTION OF T'_C	84
17	SYMMETRIC DISPLAY OF ISOLINES OF $E'_X \times 10^5$ IN THREE DIMENSIONAL OSCILLATORY SHEAR FLOW.	89
18	NON-DIMENSIONAL LONGITUDINAL DISPERSION COEFFICIENT, E'_L , AS A FUNCTION OF T'_C FOR THREE DIMENSIONAL UNIDIRECTIONAL SHEAR FLOW.	98
19	COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL DISPERSION IN THREE DIMENSIONAL UNIDIRECTIONAL SHEAR FLOW	101
20	COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL DISPERSION FOR THE ENGLISH CHANNEL, LIVERPOOL BAY, AND CUMBERLAND COAST	109
21	COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL DISPERSION FOR THE MERSEY ESTUARY.	112
22	COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL DISPERSION USING SEGALL'S DATA (19).	114

LIST OF SYMBOLS AND ABBREVIATIONS

- a - Pipe radius
- $a_n, a_m, a_{mn},$
 a'_m, a'_{lk} - Fourier coefficients
- A - Flow cross-sectional area
- A_1 - General function used by Bowden
- B - General function used in examination of conditions for which concept of dispersion is valid in oscillatory free surface flow
- B_1 - General function used by Bowden
- c - Concentration of substance
- $\bar{c}$ - Mean concentration of substance over flow cross section
- c'' - Variation of substance concentration from cross-sectional mean value
- c''_s - Separation variable representing spatially dependent portion of c''
- c''_{ND} - Non-dimensional form of c''
- cm - Centimeters
- D_r - Molecular diffusion coefficient in radial direction
- D_t - Molecular diffusion coefficient in direction transverse to principal flow axis
- e_v - Vertical molecular or eddy diffusivity used by Elder
- E_L - Longitudinal dispersion coefficient for unidirectional flow
- E_x - Longitudinal dispersion coefficient for oscillatory flow, temporally dependent
- E_∞ - Time averaged longitudinal dispersion coefficient for oscillatory flow with an infinitely large period of oscillation, used by Holley et al.

LIST OF SYMBOLS AND ABBREVIATIONS (Continued)

- E'_x - Non-dimensional time averaged longitudinal dispersion coefficient for oscillatory flow
- $\langle E_x \rangle_T$ - Longitudinal dispersion coefficient averaged over period of oscillation
- E'_L - Non-dimensional longitudinal dispersion coefficient for unidirectional flow
- f - Summation index
- f_{DW} - Darcy-Weisbach friction factor
- f_1 - General function used in examination of conditions for which concept of dispersion is valid in oscillatory free surface flow
- f_2 - General function used in examination of conditions for which concept of dispersion is valid in unidirectional free surface flow
- ft - Feet
- F - Function used to simplify u_{max} normalization of two dimensional oscillatory shear flow solution
- F_1 - Functional distribution of turbulent pipe flow used by Taylor
- g - Function used to simplify velocity solution for three dimensional oscillatory shear flow
- g_1 - General function used in examination of conditions for which concept of dispersion is valid in oscillatory free surface flow
- g_2, G - General functions used in examination of conditions for which concept of dispersion is valid in unidirectional free surface flow
- h - Water depth
- i - $\sqrt{-1}$
- I_1, I_2 - Functions used to simplify derivation of u_{max} normalization for three dimensional oscillatory shear flow solution
- j - Summation index
- k - Summation index

LIST OF SYMBOLS AND ABBREVIATIONS (Continued)

- K - Pressure gradient modulus
- K_0 - Amplitude of eddy diffusion coefficient used by Fukuoka
- K_z - Vertical coefficient of eddy diffusion
- K_y - Lateral coefficient of eddy diffusion
- $\bar{K}_z$ - Mean cross-sectional value of K_z
- $\bar{K}_y$ - Mean cross-sectional value of K_y
- ℓ - Summation index
- ℓ_s - Distance from point of maximum surface velocity to most distant bank, used by Fischer
- L - Surface excursion length
- m - Summation index
- $\dot{m}$ - Mass transport
- $\langle \dot{m} \rangle_T$ - Average mass transport over one period of oscillation
- M - Meters
- n - Summation index
- osc.- Oscillatory
- p - Summation index
- P - Pressure
- P_1 - General function used by Bowden
- q - Summation index
- q'' - Depth integrated u'' used by Fischer
- Q_1 - General function used by Bowden
- r - Cartesian coordinate in radial direction
- r_1, r_2 - Functions used to simplify presentation of E'_x for three dimensional oscillatory shear flow
- R_e - Real part of complex function

LIST OF SYMBOLS AND ABBREVIATIONS (Continued)

- R_H - Hydraulic radius
- R_1, R_2 - Functions used to simplify development of three dimensional oscillatory shear flow solution for u'' , c'' , and $\langle E_x \rangle_T$
- R_{mn}, R_{pq}, R_{lk} - Functions used to simplify development of three dimensional unidirectional shear flow solution for u'' , c'' , and E_L .
- sec - Seconds
- t - Time
- t' - Variable of integration
- T - Period of oscillation
- T_{cy} - Lateral mixing time
- T_{cz} - Vertical mixing time
- T_0 - Semi-diurnal tidal period, 44,712 sec.
- T' - Non-dimensional vertical mixing time used by Holley et al.
- T'_c - Relative mixing time
- $T'_{c\infty}$ - Relative mixing time for which two dimensional oscillatory E'_x solution is valid
- T'_y - Non-dimensional lateral mixing time
- T'_z - Non-dimensional vertical mixing time
- u - Velocity component along principal flow axis, x
- u_f - Unidirectional flow component used by Fukuoka
- u_{max} - Temporal and spatial maximum of u
- u_s - Separation variable representing spatially dependent portion of u
- u_t - Oscillatory flow component used by Fukuoka
- $\bar{u}$ - Mean value of u over flow cross section
- u'' - Variation of u from cross-sectional mean value

LIST OF SYMBOLS AND ABBREVIATIONS (Continued)

- u''_{ND} - Non-dimensional form of u''
- u''_S - Separation variable representing spatially dependent portion of u''
- u_* - Shear velocity
- U_o - Surface amplitude of unidirectional flow component used by Okubo
- unidir - Unidirectional
- V_o - Surface amplitude of oscillatory flow component used by Okubo
- w - Width of channel
- x - Cartesian coordinate directed along principal flow axis of the channel
- y - Cartesian coordinate directed in lateral direction across width of channel
- z - Cartesian coordinate directed in vertical direction over depth of channel
- z' - Variable of integration
- α - Constant
- $\alpha_1, \alpha_2, \alpha_3$ - Phase angles
- β - Shear wave parameter in two dimensional oscillatory shear flow solution
- $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ - Functions used to simplify three dimensional shear flow solutions
- ϵ_z - Coefficient of vertical eddy viscosity
- ϵ_y - Coefficient of lateral eddy viscosity
- $\bar{\epsilon}_z$ - Mean value of ϵ_z over flow cross section
- $\bar{\epsilon}_y$ - Mean value of ϵ_y over flow cross section
- θ_1, θ_2 - Functions used to simplify development of $\langle E_x \rangle_T$ for three dimensional oscillatory shear flow

LIST OF SYMBOLS AND ABBREVIATIONS (Continued)

- v - Locus of channel perimeter
- λ - Function used to simplify development of $\langle E_x \rangle_T$ for three dimensional oscillatory shear flow
- μ_{lk} - Factor used to properly account for a'_{00} , a'_{0k} , and a'_{l0} terms in three dimensional shear flow solutions.
- ξ - Cartesian coordinate directed along principal flow axis of the channel as seen by point traveling with velocity $\bar{u}$
- ρ - Fluid density
- σ - Angular frequency of oscillation, $\frac{2\pi}{T}$
- Σ - Summation symbol
- τ - Non-dimensional time used by Aris
- ϕ, ϕ' - Functions used to simplify u_{max} normalization of three dimensional shear flow solutions
- χ - General function
- ψ - Function used to simplify presentation of E'_x for three dimensional oscillatory shear flow solution

Abstract of Dissertation Presented to the Graduate Council
of the University of Florida in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy

DISPERSIVE MASS TRANSPORT IN OSCILLATORY
AND UNIDIRECTIONAL FLOWS

By

Robert Bruce Taylor

August, 1974

Chairman: Robert G. Dean

Major Department: Civil and Coastal Engineering

Four sets of boundary value problems for the equations of fluid motion and transport diffusion are solved to obtain expressions for the spatial variations of velocity and concentration in free surface flows. The four problems investigated treat unidirectional and oscillatory uniform shear flows in rectangular channels of infinite and finite width. The solutions are then used to determine longitudinal dispersive mass transport. Analytical expressions for the longitudinal dispersion coefficient associated with infinitely wide channels are presented as functions of the temporal and spatial maximum of velocity, channel depth, period of oscillation (oscillatory flow only), and depth mean vertical coefficient of eddy diffusivity. For channels of finite width, analytical expressions for the longitudinal dispersion coefficient are presented as functions of the temporal and spatial maximum of velocity, channel depth, channel width, period of oscillation (oscillatory flow only), and cross-sectional mean values of vertical and lateral coefficients of eddy diffusivity.

It is found that for infinitely wide channels the longitudinal dispersion coefficient in oscillatory flow is described by a type of resonant

interaction between the period of oscillation and the time scale of vertical mixing. The maximum longitudinal dispersive mass transport occurs when the ratio of the vertical mixing time to the period of oscillation equals 1.58. This functional behavior is shown to be unique and distinctly different from the dispersive process in unidirectional flow.

For rectangular channels of finite width it is shown that the dispersive mass transport process is symmetric, such that a channel geometry skewed in width produces the same dispersive mass transport as a channel geometry skewed in depth, provided that specific requirements regarding the maximum velocity, period of oscillation, and vertical and lateral mixing times are satisfied.

The longitudinal dispersion coefficient for oscillatory flow in rectangular channels of finite width is presented as a family of resonance curves described by the resonant interaction of the period of oscillation with the vertical and lateral mixing times.

For unidirectional flow in channels of finite width it is found that the predicted non-dimensional longitudinal dispersion coefficient increases without bound as the skewness of the channel geometry is correspondingly increased. Thus, for unidirectional flow the three dimensional solution does not collapse in the limit to the two dimensional case as is found for oscillatory flow.

Predicted values for the longitudinal dispersion coefficient are compared with values based upon field and laboratory data. The comparison shows that when reliable measurements are available for all of the functional parameters included in the predictive solutions the agreement between observed and predicted values is good.

CHAPTER I INTRODUCTION

I-A General Background

A growing awareness in recent years of the need for a rational approach to the planned use and development of coastal regions has significantly contributed to a concern for maintaining an acceptable level of water quality within our river, estuarine, and bay system waters. To achieve this a fundamentally sound understanding of the physical processes which govern the mass transport characteristics of these waters is essential.

The mass transport of a substance as it applies here may be defined as "the time averaged net transfer of substance along the principal flow axis of the watercourse or channel." With the aid of this definition one can intuitively reason that the mass transport mechanism for a conservative substance is governed by the kinematic description of the flow field and the distribution of the substance within the flow field. Unfortunately, the complexities associated with turbulent shear flows in natural watercourses have thus far precluded a complete analytical description of either the flow kinematics or the distribution of the substance. However, the analytical treatment of longitudinal mass transport in steady uniform axisymmetric shear flows by Taylor (1,2) provided for the first time a reasonable method for predicting these phenomena.

In his analysis Taylor made the distinction between three mechanisms of longitudinal mass transport:

- (1) Convective transport of the cross-sectional mean concentration of substance by the mean cross-sectional velocity,

(2) Convective transport due to the correlation over the cross section of the spatial variations from the mean of velocity and concentration,

(3) Diffusion of the substance along the principal flow axis (molecular diffusion in laminar flow and eddy diffusion in turbulent flow).

The first of these mechanisms has long been recognized as a mass transport phenomenon and is the most amenable of the three to an analytical treatment. In the absence of diffusive transport due either to molecular or turbulent motion this mechanism may be visualized as the transport along the principal flow axis of a slug of substance uniformly mixed over the flow cross section and undistorted by boundary shear. The third mechanism, that due to longitudinal diffusive transport, was shown by Taylor to be less than 1 per cent of the second which he defined as dispersive mass transport. It is this second mechanism, longitudinal dispersive mass transport, which is the subject of the present work.

Following Taylor's initial formulation of the dispersion mechanism investigators began to extend his analysis to unidirectional two dimensional open channel flows for infinitely wide channels and subsequently to the more realistic three dimensional case of unidirectional flow in a channel of finite width. More recently, the emphasis of the research effort has shifted from steady unidirectional flow to oscillatory flow in an effort to apply the knowledge obtained from the former to develop a predictive capability for dispersive mass transport for the latter. A brief summary of previous efforts follows.

I-B Discussion of Previous Work on Dispersive Mass Transport

I-B.1 Unidirectional Flow

As mentioned previously, Taylor (1) was the first investigator to define the mechanism of dispersive mass transport. In his paper Taylor considered the case of a passive, conservative substance transported by a laminar steady flow in a circular tube. A passive substance is one which is present in small enough concentrations so as not to alter either the velocity flow field or the magnitude of the diffusion coefficient; a conservative substance is one which has a fixed total mass present for all time in the system being considered. All substances referred to in the present work should be considered to be both passive and conservative. To begin his analysis, Taylor transformed the Eulerian form of the transport diffusion equation to one with the origin moving with the cross-sectional mean velocity by defining

$$\xi = x - \bar{u}t \quad (\text{I-1})$$

$$u = \bar{u} + u'' \quad (\text{I-2})$$

where $\bar{u}$ is the cross-sectional mean velocity, u'' is the variation from the mean of the velocity in the radial direction due to the shear profile, and x is the coordinate along the axis of the tube positive in the direction of flow. Neglecting molecular diffusion along the axis of the tube he was then able to write

$$\left(\frac{\partial c}{\partial t}\right)_{\xi} + u'' \frac{\partial c}{\partial \xi} = \frac{D}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c}{\partial r}\right) \quad (\text{I-3})$$

where,

$$\begin{aligned} \left(\frac{\partial c}{\partial t}\right)_{\xi} &= \text{time rate of change of concentration as seen from moving} \\ &\quad \text{coordinate system} \\ &= \frac{\partial c}{\partial t} + \bar{u} \frac{\partial c}{\partial x} \end{aligned}$$

D_r = molecular diffusion coefficient in radial direction.

If (I-3) is integrated over the cross-sectional area of the tube and the concentration, c , decomposed into its cross-sectional mean and variation component, by the expression

$$c = \bar{c} + c'' \quad (I-4)$$

then (I-3) reduces to

$$\left(\frac{\partial \bar{c}}{\partial t}\right)_\xi = - \frac{\partial}{\partial \xi} (\overline{u''c''}) \quad (I-5)$$

For standardization throughout the text overbars shall be considered to represent cross-sectional averaging whereas the symbol $\langle \rangle_T$ shall be considered to represent temporal averaging over the periodic interval T . The form of (I-5) led Taylor to define the convective transport shown as a Fickian flux which he called dispersion, or

$$E_L = - \frac{1}{A \frac{\partial \bar{c}}{\partial \xi}} \iint_A u''c'' dA \quad (I-6)$$

where E_L is the longitudinal dispersion coefficient for steady unidirectional flow and A is the flow cross-sectional area. As indicated by (I-6) the heart of Taylor's analysis lies in the determination of the spatial variations of velocity and concentration, u'' and c'' respectively. The velocity variation was readily determined from the parabolic velocity profile. To determine c'' Taylor returned to (I-3), decomposed c by (I-4) and made the following assumptions:

- (1) Steady state conditions exist with respect to the moving coordinate system, i.e. $\left(\frac{\partial \bar{c}}{\partial t}\right)_\xi = 0$.
- (2) c'' is not a function of ξ .
- (3) $\frac{\partial \bar{c}}{\partial \xi}$ is constant.

These assumptions reduced (I-3) to

$$u'' \frac{\partial \bar{c}}{\partial \xi} = \frac{D_r}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c''}{\partial r} \right) \quad (\text{I-7})$$

which Taylor then solved for c'' , performed the integration of $u''c''$ in (I-6) and produced the following expression for the longitudinal dispersion coefficient

$$E_L = \frac{a^2 u_{\max}^2}{192 D_r} \quad (\text{I-8})$$

where a is the radius of the tube, and $u_{\max}$ is the centerline velocity. It is important to note that the dispersion coefficient varies directly with the ratio a^2/D_r which represents the time scale of mixing of the substance over the flow cross section due to transverse diffusivity. Using the definition given in (I-6) for E_L , (I-5) may then be written as

$$\left(\frac{\partial \bar{c}}{\partial t} \right)_{\xi} = E_L \frac{\partial^2 \bar{c}}{\partial \xi^2} \quad (\text{I-9})$$

Equation (I-9), with E_L given by (I-8), shows that the cross-sectional mean concentration of a substance is effectively diffused longitudinally about a point traveling with the mean velocity by the combined effect of the velocity shear profile and the transverse diffusivity. Taylor correctly cautioned, however, that in applying (I-9) the time scale of convective transport associated with u'' must be significantly greater than the time scale of cross-sectional mixing due to transverse diffusivity. A further discussion of the assumptions made by Taylor and the application of dispersion coefficients obtained in this manner will be presented in Chapter II.

Having looked at laminar flows Taylor (2) extended his analysis to axisymmetric turbulent shear flow in a pipe. For u'' Taylor used an experimentally verified velocity distribution of the form

$$\frac{u_{\max} - u(r)}{u_*} = F_1(r) \quad (I-10)$$

where $u_{\max}$ is the centerline velocity, u_* the shear or friction velocity, and $F_1(r)$ an experimentally determined function. Then by assuming Reynolds analogy which states that the transfer of mass, heat, and momentum by turbulence are exactly analogous, and the same conditions required to obtain (I-7) he obtained a solution for c'' and numerically performed the integration of $u''c''$ shown in (I-6). The result was the following expression for the longitudinal dispersion coefficient in turbulent pipe flow

$$E_L = 10.06 a u_* \quad (I-11)$$

where a is the radius of the pipe. By assuming isotropy of the turbulent fluctuations he was then able to estimate the additional longitudinal transport due to eddy diffusion along the flow axis. Adding this to (I-11), the corrected dispersion coefficient becomes

$$E_L = 10.1 a u_* \quad (I-12)$$

Shortly after Taylor's discovery and formulation of dispersive mass transport, Aris (3) presented a very elegant analysis which supported Taylor's theory for unidirectional steady flows and rigorously defined the conditions under which Taylor's assumptions were valid. Rather than obtaining solutions for u'' and c'' to compute the dispersive mass transport, Aris chose to apply a moment analysis of the longitudinal distribution

of the cross-sectional mean concentration of substance about a point moving with the mean velocity. The significance of Aris' analysis lies in its generality, and the insight that it yields for looking at the precise conditions under which an ordered correlation of $u''c''$ over the cross-sectional flow area may be represented by the implied stationary random process of a Fickian flux. Aris' development begins with the basic Eulerian form of the transport diffusion equation for an arbitrary flow geometry and an arbitrary initial distribution of substance. He then proceeds to develop expressions for the moments up through the third of the cross-sectional mean concentration about ξ with no assumptions regarding steady state, or the functional form of the concentration c . This generality allows him to obtain time dependent solutions which show that after an initial transient period, steady state conditions will be asymptotically approached in which the cross-sectional mean concentration, $\bar{c}$, disperses about a point traveling at the mean speed of the flow in a Gaussian manner and that the rate of growth of the variance is constant with time. In other words the critical assumption in Taylor's analysis is the one concerning steady state conditions with respect to the moving coordinate system, i.e. $(\frac{\partial c}{\partial t})_{\xi} = 0$, which by Aris' analysis must be satisfied if the definition of the convective mass transport of a substance as a Fickian flux about the mean speed of flow is to hold. The dimensionless time parameter governing the dispersive process is defined by Aris as

$$\tau = \frac{D_t t}{a^2} \quad (I-13)$$

where t represents time; D_t is the molecular diffusion coefficient in the transverse direction; and a is a characteristic length of the flow cross section. The ratio a^2/D_t , as noted previously, represents the time scale

required for the substance to mix over the flow cross section due to transverse diffusion. Thus, steady state conditions are approached for large t , or when the convective flow time is large compared to the time required for cross-sectional mixing. This supports Taylor's comments regarding the application of dispersion coefficients to predict concentration distributions. In Chapter II it will be shown how the assumption of steady state conditions supports Taylor's other assumption regarding the functional dependence of c for both steady unidirectional and oscillatory flows. Aris' method of analysis has been used by other investigators to obtain expressions for the dispersion coefficient and for brevity later in the text will be referred to as the "method of moments." Although it remains a very useful method for determining expressions for longitudinal dispersion coefficients it does not provide the physical insight into the kinematic flow field and cross-sectional distribution of substance that Taylor's method does.

Further support of the general applicability of Taylor's dispersion analysis to different flow geometries was given by Elder (4) who applied Taylor's analytical technique to obtain a predictive expression for the dispersion coefficient in two dimensional unidirectional shear flows in infinitely wide open channels. In his analysis Elder obtains the following expression for the longitudinal dispersion coefficient:

$$E_L = \frac{1}{h} \int_0^h u'' \int_0^z \frac{1}{e_v} \int_0^z u'' dz' dz' dz' \quad (I-13)$$

where z' is a variable of integration; z is the vertical coordinate; h is the depth of flow; and e_v is either the vertical molecular diffusivity or the vertical eddy diffusivity depending upon whether the flow is laminar or turbulent. This expression indicates, in a manner similar to (I-8),

that in either laminar or turbulent unidirectional shear flow the longitudinal dispersion varies directly with the time scale of cross-sectional mixing. To obtain an expression for E_L Elder assumed the velocity distribution to be logarithmic which upon substitution into (I-13) and correcting for longitudinal eddy diffusion as done by Taylor, yields

$$E_L = 5.93 u_* h \quad (I-14)$$

where u_* is the shear velocity.

The importance of the flow cross section geometry on the velocity shear profile, transverse diffusivity, and ultimately the longitudinal dispersion is demonstrated by (I-12) and (I-14) which if expressed in terms of the hydraulic radius, R_H , for the respective flows yield values of $E_L/R_H u_*$ of 20.2 for the pipe and 5.9 for the infinitely wide channel. Bowden (5) developed expressions for E_L using several different velocity and eddy diffusivity distributions with depth and showed that $E_L/u_* h$ ranged from 5.9 to 25 for the cases analyzed. Thus, the particular velocity shear profile used along with the cross-sectional eddy diffusivity have a significant effect on the magnitude of the predicted longitudinal dispersion coefficient.

While the works of Taylor, Aris, and Elder proved the existence of an effective dispersive mass transport mechanism and there was reasonable agreement between laboratory experimental data for pipe and infinitely wide open channel flow geometries, other investigators were finding that observed values of the longitudinal dispersion coefficient for natural rivers and streams were considerably higher than those predicted by theory. As reported by Fischer (6) observed values for E_L in natural watercourses ranged from 50 to 700 hu_* as compared with the 5.9 to 25 hu_* range predicted

by Elder and Bowden. To account for this increased effect Fischer concluded that the dominant mechanism in longitudinal dispersion was the interaction of the velocity shear profile and turbulent mixing time scales across the finite width of the channel as opposed to the vertical variations treated by the previous investigators for infinitely wide channels. The effect of asymmetrical flow cross sections on longitudinal dispersion was first examined by Aris (3) who showed that the dispersion in a circular tube is less than that in an elliptical one of the same area. This in addition to the fact that all real watercourses have a finite width and that width-to-depth ratios for natural channels are usually significantly greater than unity makes Fischer's hypothesis very reasonable. For wide channels the transverse mixing time, ℓ_s^2/K_y , over some characteristic ℓ_s would tend to be greater than the corresponding vertical mixing time scale, h^2/K_z . Fischer thus argued that the velocity shear profile across the channel would have a greater longitudinal dispersive effect since an increased mixing time would produce larger values of the spatial correlation $\overline{u''c''}$. Using this as the basis for his analysis Fischer applied Taylor's technique to a rectangular coordinate system and assumed that vertical variations in c'' were negligible compared to the transverse variations to obtain

$$u'' \frac{\partial \bar{c}}{\partial x} = \frac{\partial}{\partial y} \left(K_y \frac{\partial c''}{\partial y} \right) \quad (I-15)$$

where y is the coordinate axis across the channel.

Fischer then integrated (I-15) over the cross-sectional area of the channel to obtain an expression for c'' which he then correlated with u'' for the dispersive mass transport and using Taylor's definition of a Fickian flux obtained a longitudinal dispersion coefficient of

$$E_L = -\frac{1}{A} \int_0^w q''(y) dy \int_0^y \frac{1}{K_y h(y)} dy \int_0^y q''(y) dy \quad (I-16)$$

where the depth, h , is a function of y ; K_y is the transverse eddy diffusivity; w is the width of the channel; and $q''(y)$ is the depth integrated flow defined by

$$q''(y) = \int_0^{h(y)} u''(y, z) dz \quad (I-17)$$

To obtain a more useful expression for E_L Fischer defines a Lagrangian time scale for cross-sectional mixing which he obtains through an extension of Taylor's work on diffusion by continuous movements (7), and relates it to the Eulerian time scale for cross-sectional mixing previously discussed. Then, using Elder's (4) experimental determination that the transverse eddy diffusivity in an infinitely wide channel with a steady unidirectional flow can be expressed as

$$K_y = 0.23 hu_* \quad (I-18)$$

Fischer arrives at an alternate expression for E_L given by

$$E_L = 0.3 \frac{\ell_s^2}{u_*^2} \frac{u_*^2}{R_H u_*} \quad (I-19)$$

where ℓ_s in a natural channel is the distance from the point of maximum surface velocity to the most distant bank. To verify his theory Fischer conducted a detailed set of laboratory experiments. Using width-to-depth ratios ranging from 9 to 15.7 Fischer obtained good agreement between predicted and observed values of E_L using (I-16). However, (I-19) over-predicted in each case, with a maximum error of 75 per cent.

Fischer (8) also applied his analysis to data obtained by other investigators

in 6 natural watercourses including Copper Creek, Va. and Clinch River, Tenn. (2 locations each), Powell River, Tenn., and Coachella Canal, California. Width-to-depth ratios for these sites range from 15 to 62 with most being less than 40. To carry out his analysis Fischer numerically integrated (I-16) using measured data and found that in uniform channels the predicted values for E_L were within 30 per cent of the observed values whereas in non-uniform channels the predicted value varied from the observed by as much as a factor of 4.

Thus, while Fischer's hypothesis regarding the dominance of lateral effects on longitudinal dispersion in unidirectional flows yields reasonable results his analysis provides very little insight into the physics of the three dimensional problem. Both (I-16) and (I-19) make no attempt to describe the velocity distribution over the flow cross section thereby requiring detailed measurements for their application. Moreover, by neglecting the vertical shear effects it remains impossible to obtain from his analysis the conditions for which either the vertical or lateral effects would be the dominant mechanism for longitudinal dispersion.

I-B.2 Oscillatory Flow

The analysis of dispersive mass transport in oscillatory flow is more complex than the corresponding problem in steady unidirectional flow because of the unsteady nature of the governing equations of motion and transport diffusion. Since, however, many important water quality problems involve the predicted distribution and transport of substances within tidal waters the analysis of dispersive processes in oscillatory flows has attracted considerable interest. There is in fact a reasonable doubt that one is justified in representing the convective mass transport $u''c''$

as a Fickian flux in an oscillatory flow. However, it will be shown in Chapter II that under the assumptions of periodicity and uniform flow the analyses of Taylor and Aris apply.

Bowden (5) was the first investigator to look at longitudinal dispersive mass transport in oscillatory flow. Bowden generally followed Taylor's original analytical technique and began by assuming a monochromatic flow in an infinitely wide channel in which u'' can be expressed as

$$u''(z,t) = A_1(z) \cos \sigma t + B_1(z) \sin \sigma t \quad (I-20)$$

He likewise assumed that c'' is simple harmonic in time and of the form

$$c''(z,t) = P_1(z) \cos \sigma t + Q_1(z) \sin \sigma t \quad (I-21)$$

conditional upon satisfying the transport diffusion equation expressed as

$$\left(\frac{\partial c''}{\partial t}\right)_{\xi} + u'' \frac{\partial \bar{c}}{\partial \xi} = \frac{\partial}{\partial z} \left(K_z \frac{\partial c''}{\partial z} \right) \quad (I-22)$$

This is a very proper formulation of the problem, allowing for the periodic variation of u'' and c'' with time and the existence of temporal phase shifts for u'' and c'' as a function of position within the water column. Unfortunately, Bowden proceeded by means of his assumptions to reduce the problem to one which applies only to the case for which the period of oscillation is infinitely long, $\left(\frac{\partial c''}{\partial t}\right)_{\xi} = 0$, and for which the temporal phase shift as a function of position within the water column has been eliminated. For these conditions Bowden found that the longitudinal dispersion coefficient in an oscillatory flow is one-half the value of the same coefficient for a corresponding unidirectional flow having the same surface velocity, the same shear profile, and the same vertical eddy diffusivity.

Holley and Harleman (9) treated the case of oscillatory turbulent pipe flow and obtained an expression for the longitudinal oscillatory flow dispersion coefficient, E_x , using Aris' method of moments. In their analysis they assumed that at each instant in time the lateral distribution of velocity and eddy diffusivity are the same as they would be if the flow were steady with the same cross-sectional mean velocity. This assumption effectively removes the oscillatory nature of the flow and reduces the problem to essentially the same one solved by Aris (3) and Taylor (2). Thus, it is not surprising that Holley and Harleman predicted a longitudinal dispersion coefficient of

$$E_x(t) = 10.1a u_* (t) \quad (I-23)$$

where, as before, a is the radius of the pipe, and $u_* (t)$ is the friction velocity now periodic in time. Predicted values of E_x were compared with observations made from a series of laboratory experiments in which an oscillatory flow was produced in a 1-1/2 inch pipe by a piston generator. Superimposed upon the oscillating flow was a small unidirectional flow to simulate the net fresh water discharge of an estuarine type flow. The experimental data, however, showed that due to the small magnitude of the unidirectional flow compared to the velocity amplitude of the oscillatory flow, its effect on longitudinal dispersion could be neglected. The data also showed that (I-23) generally underpredicted the dispersion coefficient with better agreement between predicted and observed values coming with higher Reynolds number flows. The authors attributed this behavior to laminar sublayer effects.

Okubo (10) obtained a less restrictive solution which for the first time demonstrated that in a two dimensional oscillatory shear flow the

period of oscillation and the time scale of vertical mixing played an important role in the dispersive process. Okubo confined his analysis to vertically bounded and unbounded flows with no transverse shear. Only the former will be discussed. He assumed a linear, monochromatic flow superimposed upon a linear unidirectional flow, or

$$u(z,t) = U_0 \left(1 - \frac{z}{h}\right) + V_0 \left(1 - \frac{z}{h}\right) \sin \sigma t \quad (\text{I-24})$$

where U_0 and V_0 are the surface amplitudes of the unidirectional and oscillatory flow, and σ is the angular frequency of oscillation. Okubo then assumed a constant eddy diffusivity both in the direction of flow and over the depth, and zero diffusive flux conditions at the surface and bottom. Using the method of moments as applied by Aris and the assumptions discussed he obtained an expression for the longitudinal dispersion coefficient, E_x , functionally expressed as

$$E_x = \chi \left(U_0, V_0, h, T, \frac{h^2}{K_z} \right) \quad (\text{I-25})$$

where T is the period of oscillation, and $\frac{h^2}{K_z}$ the time scale of vertical mixing. To interpret his results Okubo considered two extreme cases.

(1) $\frac{h^2}{K_z} \gg T$ - Neglecting longitudinal eddy diffusion his solution becomes

$$E_x = \underbrace{\frac{U_0^2}{120} \left(\frac{h^2}{K_z} \right)}_{\text{unidirectional}} + \underbrace{\frac{V_0^2}{4\pi^2} \left(\frac{T^2 K_z}{h^2} \right)}_{\text{oscillatory}} \quad (\text{I-26})$$

From this Okubo correctly concluded that for situations in which the vertical mixing time is large compared with the period of oscillation the contribution to the longitudinal dispersive mass transport due to the oscillatory flow is small compared with the contribution due to the unidirectional flow. However, (I-26) also suggests that the functional behavior of dispersion

in an oscillatory flow is significantly different from the same process in unidirectional flow. One of the principal findings of the present work concerns this point and will be treated in depth in Chapters III and IV.

(2) $T \gg \frac{h^2}{K_z}$ - Again neglecting longitudinal eddy diffusion Okubo's result may be written as

$$E_x = \frac{U_0^2}{120} \left(\frac{h^2}{K_z} \right) + \frac{V_0^2}{236} \left(\frac{h^2}{K_z} \right) \quad (I-27)$$

This case is comparable to Bowden's analysis for a very long period of oscillation and supports his finding that $E_x \rightarrow 1/2 E_L$ for large T .

Holley et al. (11) successfully applied Taylor's original method of analysis to obtain an analytical expression for $\langle E_x \rangle_T$ in a two dimensional periodic, uniform shear flow. For their analysis they assumed the spatial variation component of velocity to be

$$u''(z,t) = \alpha z \sin \sigma t \quad (I-28)$$

where α is a constant. This is identical to the linear oscillatory flow profile assumed by Okubo. This expression for u'' is then used to force the following form of the transport diffusion equation to obtain a solution for $c''(z,t)$

$$\left(\frac{\partial c''}{\partial t} \right)_\xi - K_z \frac{\partial^2 c''}{\partial z^2} = -u'' \frac{\partial \bar{c}}{\partial \xi} \quad (I-29)$$

The solutions obtained for u'' and c'' are then used to determine the depth integrated time mean convective mass transport with respect to the depth mean velocity. The resulting expression for the mean longitudinal dispersion coefficient over the period of oscillation is

$$\langle E_x \rangle_T = \frac{\alpha^2 T^2 h^4}{\pi^4 K_z} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 \left\{ \left[\frac{\pi}{2} (2n-1)^2 T' \right]^2 + 1 \right\}} \quad (I-30)$$

where $T' = \frac{TK_z}{h^2}$. Holley et al. then proceed to relate $\langle E_x \rangle_T$ in an oscillating flow to the dispersive process in an oscillating flow of infinite period with the same velocity profile as given by (I-28). Solving for c'' using the steady state form of (I-29) and proceeding in the usual manner they obtain an expression for $\langle E_x \rangle_T$ which they call E_∞ . It is shown that E_∞ is one-half the corresponding value for E_L in agreement with Bowden (5), and is subsequently used to compare $\langle E_x \rangle_T$ with the corresponding steady unidirectional flow coefficient. Apparently the investigators' primary goal was not to describe longitudinal dispersion in an oscillatory flow as a separate distinct physical process but rather to describe it in terms of a "corresponding" process in a unidirectional flow. To do this the solution for $\langle E_x \rangle_T$ given by (I-30) was normalized by E_∞ and the result plotted against the non-dimensional time, $T' = \frac{TK_z}{h^2}$, suggested by Okubo's (10) results. Numerical simulations were also carried out for other velocity profiles in which the velocity was again assumed to be temporally in phase over depth during the period of oscillation. Based upon their analytical and numerical results Holley et al. concluded that:

(1) $T' > 1$ - The longitudinal dispersion coefficient in an oscillatory flow is independent of T' and equal to one-half the value of the dispersion coefficient in a "corresponding" unidirectional flow.

(2) $T' < 0.1$ - The longitudinal dispersion coefficient in an oscillatory flow rapidly becomes insignificant compared to the dispersion coefficient in a "corresponding" unidirectional flow and is functionally described by

$$\langle E_x \rangle_T \sim T'^2 \quad (I-31)$$

These results have been subsequently applied by Fischer (12, 13), Awaya (14), and Fischer and Holley (15) in analyses of dispersion in oscillatory

flows. Fischer (12) attributes the functional behavior of oscillatory flow dispersion to a temporal phase shift between u'' and c'' as a function of T' . However, it will be shown in the present work that while these conclusions are applicable for the non-dimensional parameter $\frac{\langle E_x \rangle T}{E_\infty}$, they are incorrect and misleading in their description of the longitudinal dispersive mass transport process in oscillatory flow.

Fukuoka (16) obtained analytical expressions for E_x from several cases of two dimensional and axisymmetric oscillatory shear flows using the method of moments. For his analysis he assumed a velocity of the general form

$$u = u_s(z) + u_t(z) \sin \sigma t \quad (I-32)$$

similar to Okubo (10), implying once again no temporal phase shift of the velocity over the flow cross section. In specifying the forms of $u_s(z)$ and $u_t(z)$, Fukuoka assumed that the spatial variation at each instant of time is the same as that for an equivalent unidirectional flow. Specific cases treated in depth are:

- (1) linear velocity profile over depth with constant K_z
- (2) parabolic velocity profile for axisymmetric flow with constant

$$K_z$$

- (3) linear velocity profile over depth with $K_z = K_0 \left| \sin \sigma t \right|$.

In interpreting his results Fukuoka follows Holley et al. by plotting $\frac{E_x}{E_\infty}$ vs. T' and arrives at the same conclusion regarding the functional behavior of dispersion in oscillatory flows. For Case 3 he shows analytically that the dispersion coefficient corresponding to a time varying eddy diffusivity differs by a factor of $8/\pi^2$ from one that is considered to be constant, provided the constant K_z is $\frac{2}{\pi} K_0$. This small effect of a periodic eddy diffusivity supports results obtained numerically by Holley et al.

It is interesting to note that none of the analyses dealing with oscillatory flow discussed thus far have attempted to solve analytically the velocity distribution over the flow cross section. Moreover, it has been further assumed that the velocity profile at each instant in time is the same as an equivalent unidirectional flow profile and that no temporal phase shift as a function of spatial position exists. These assumptions are particularly unrealistic when one considers the definition of dispersive mass transport as defined by Taylor, i.e. $\iint_A u''c'' dA$. By restricting the form of u'' in this manner one is also restricting the form of c'' since by (I-29) it is seen that the distribution of substance is in part forced by the convective flux. Thus, any artificiality regarding the assumed form of u'' appears in the dispersion coefficient through both u'' and c'' . The significantly different nature of a truly oscillatory flow as compared with a unidirectional one is illustrated by Schlichting (17) in his treatment of the fluid motion above an oscillating flat plate, and Lamb (18) in his treatment on the effect of bottom friction on long waves in two dimensional shear flow. Although the forcing functions in these two cases are different (oscillating boundary shear in Schlichting as opposed to an oscillating pressure gradient in Lamb), the result of a shear wave propagating through the fluid in a direction normal to the axis of flow is similar, causing a continuously changing phase shift of the flow over the cross-sectional area.

Awaya (14), Segall (19), and Segall and Gidlund (20) have obtained analytical expressions for the longitudinal dispersion coefficient in oscillatory flow situations using realistic velocities obtained directly from solutions to the appropriate forms of the equations of motion. Awaya addresses the case of an oscillatory laminar flow in a circular cross section for which he obtains an expression for $\langle E_x \rangle_T$ using Taylor's technique, and

normalizes it by E_∞ after Holley et al. His results are similar to those of Holley et al. and his interpretation the same. Segall and Segall and Gidlund use a simplified version of Lamb's previously mentioned solution for u corresponding to small values of the vertical eddy viscosity, ϵ_z , which they apply to the transport diffusion equation and solve using the method of moments. In both of these analyses the interpretation of the functional behavior of the dispersive process in oscillatory flows is the same as that of previous investigators although the matter is not pursued in any detail.

I-C Description and Scope of the Present Work

In the chapters that follow four sets of boundary value problems will be solved yielding analytical solutions for u'' , c'' , E_L , and $\langle E_x \rangle_T$ for the following cases:

- (1) Steady unidirectional flow in an infinitely wide rectangular channel, ϵ_z and K_z constant
- (2) Monochromatic periodic flow in an infinitely wide rectangular channel, ϵ_z and K_z constant
- (3) Steady unidirectional flow in a rectangular channel of width w ; ϵ_z , ϵ_y , K_z , and K_y constant
- (4) Monochromatic periodic flow in a rectangular channel of width w ; ϵ_z , ϵ_y , K_z , and K_y constant.

These solutions deal exclusively with dispersive mass transport in homogeneous fluids and uniform or nearly uniform flows.

In Chapter II a discussion is presented of the conditions under which Taylor's definition of dispersive mass transport is valid for unidirectional and oscillatory flows. Chapter III treats Cases 1 and 2 above yielding solutions for E_L and $\langle E_x \rangle_T$ in two dimensional shear flows while Chapter IV extends the analysis to three dimensional shear flows through Cases 3 and 4. In Chapter V predicted values of E_L and $\langle E_x \rangle_T$ using expressions developed in Chapters III and IV are compared with available data. Chapter VI presents a summary of the present work and some conclusions drawn from it. The mathematical details associated with portions of the boundary value problem solutions are reserved for Appendices A, B, and C.

CHAPTER II CONSIDERATIONS FOR APPLICATION OF TAYLOR'S DISPERSION ANALYSIS TO FREE SURFACE FLOWS

II-A General

In order to provide a solid base upon which the analyses presented in Chapters III and IV are built, it is necessary to set forth at this juncture a detailed development of Taylor's dispersion analysis as it applies to both unidirectional and oscillatory free surface flows. By doing this it is hoped that all assumptions made in obtaining solutions for u'' , c'' , E_x and E_L , and the conditions under which a dispersion coefficient may be defined and applied consistent with Aris (3) and Taylor (1,2), will be clearly understood.

Consider a three dimensional shear flow, either unidirectional or oscillatory, in a rectangular channel of width w and depth h . Let the coordinate system be selected as shown in Figure 1 with the x coordinate taken along the longitudinal axis of the channel denoting the direction of the

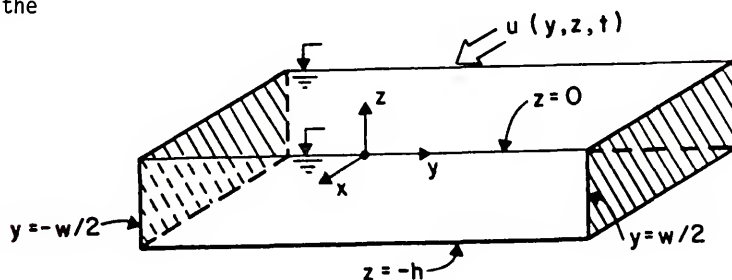


FIGURE 1 DEFINITION SKETCH

velocity component, $u(y,z,t)$; the y coordinate denoting lateral position across the channel; and the z coordinate acting positive upward from the free surface. To begin the analysis it shall be assumed that:

- (1) Longitudinal mass transport due to eddy diffusion along the flow axis is small compared to convective transport mechanisms.
- (2) Eddy diffusivities K_y and K_z can be represented by their mean values over the flow cross section.
- (3) Uniform flow conditions along the principal axis of the channel exist as indicated by the functional notation for u .

Applying these assumptions to the channel shown in Figure 1 the general form of the transport diffusion equation for a conservative substance may be written as

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = \bar{K}_y \frac{\partial^2 c}{\partial y^2} + \bar{K}_z \frac{\partial^2 c}{\partial z^2} \quad (\text{II-1})$$

After Taylor (1,2), transform (II-1) to a coordinate system traveling with the cross-sectional mean velocity using

$$u = \bar{u} + u'' \quad (\text{II-2})$$

$$\xi = x - \bar{u}t \quad (\text{II-3})$$

so that the functional transformation of the concentration c is given by

$$c(x, y, z, t) \rightarrow c(\xi(x, t), y, z, t).$$

The transformation relationships are then

$$\frac{\partial c}{\partial t} = \left(\frac{\partial c}{\partial t} \right)_{\xi} + \frac{\partial c}{\partial \xi} \frac{\partial \xi}{\partial t}$$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \xi} \cdot \frac{\partial \xi}{\partial x}$$

$$\frac{\partial c}{\partial y} \text{ and } \frac{\partial c}{\partial z} \text{ unchanged}$$

Using (II-3),

$$\frac{\partial c}{\partial t} = \left(\frac{\partial c}{\partial t}\right)_{\xi} - \bar{u} \frac{\partial c}{\partial \xi} \quad (\text{II-4})$$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \xi} \quad (\text{II-5})$$

Equations (II-2), (II-4), and (II-5) are then substituted into (II-1) to yield

$$\left(\frac{\partial c}{\partial t}\right)_{\xi} + u'' \frac{\partial c}{\partial \xi} = \bar{K}_y \frac{\partial^2 c}{\partial y^2} + \bar{K}_z \frac{\partial^2 c}{\partial z^2} \quad (\text{II-6})$$

Next, decompose c into its cross-sectional mean and variation components by the expression

$$c = \bar{c} + c'' \quad (\text{II-7})$$

and substitute into (II-6). For uniform flow this produces

$$\left(\frac{\partial \bar{c}}{\partial t}\right)_{\xi} + \left(\frac{\partial c''}{\partial t}\right)_{\xi} - \bar{K}_y \frac{\partial^2 c''}{\partial y^2} - \bar{K}_z \frac{\partial^2 c''}{\partial z^2} = - \frac{\partial}{\partial \xi} (u'' c'') - u'' \frac{\partial \bar{c}}{\partial \xi} \quad (\text{II-8})$$

which is the form of the transport diffusion equation as seen from a coordinate system traveling with the mean cross-sectional velocity.

Proceeding with the analysis, (II-8) is integrated over the cross-sectional area,

$$\begin{aligned} & \int_0^h \int_{-w/2}^{w/2} \left(\frac{\partial \bar{c}}{\partial t}\right)_{\xi} dydz + \int_0^h \int_{-w/2}^{w/2} \left(\frac{\partial c''}{\partial t}\right)_{\xi} dydz - \bar{K}_y \int_0^h \int_{-w/2}^{w/2} \frac{\partial^2 c''}{\partial y^2} dydz \\ & - \bar{K}_z \int_0^h \int_{-w/2}^{w/2} \frac{\partial^2 c''}{\partial z^2} dydz = - \int_0^h \int_{-w/2}^{w/2} \frac{\partial}{\partial \xi} (u'' c'') dydz - \int_0^h \int_{-w/2}^{w/2} u'' \frac{\partial \bar{c}}{\partial \xi} dydz \quad (\text{II-9}) \end{aligned}$$

It is next assumed that:

- (1) The time rate of change of the depth is small.

(2) There is no turbulent diffusion of substance across the enclosing boundaries of the flow cross section.

To apply the condition of zero diffusive flux across the boundaries of the channel it is helpful to use Green's Theorem in the plane which for the third and fourth terms on the left hand side of (II-9) states

$$\iint_A \left\{ \frac{\partial}{\partial y} \left(\bar{k}_y \frac{\partial c''}{\partial y} \right) + \frac{\partial}{\partial z} \left(\bar{k}_z \frac{\partial c''}{\partial z} \right) \right\} dA = - \oint_U \left\{ \bar{k}_y \frac{\partial c''}{\partial y} dz + \bar{k}_z \frac{\partial c''}{\partial z} dy \right\} \quad (\text{II-10})$$

where U is the perimeter of the area A . Since by assumption (2) it is required that there be no diffusive flux across U , the left hand side of (II-10) is equal to zero. Using this result, assumption (1), and noting that

$$\iint_A u'' dA = \iint_A c'' dA = 0$$

by definition, (II-9) is reduced to

$$\left(\frac{\partial \bar{c}}{\partial t} \right)_\xi = - \frac{\partial}{\partial \xi} (\overline{u'' c''}) \quad (\text{II-11})$$

As noted in I.B.1 the form of (II-11) induced Taylor to define the spatially averaged convective flux $\overline{u'' c''}$ as a Fickian flux thus transforming (II-11) to the one dimensional heat equation form

$$\left(\frac{\partial \bar{c}}{\partial t} \right)_\xi = E_L \frac{\partial^2 \bar{c}}{\partial \xi^2} \quad (\text{II-12})$$

where,

$$E_L = \frac{\overline{u'' c''}}{\left(-\frac{\partial \bar{c}}{\partial \xi} \right)} \quad (\text{II-13})$$

Referring again to I.B.1, Aris showed that (II-12) and (II-13) are valid formulations if the following necessary and sufficient conditions are satisfied: (1) steady uniform flow; and (2) steady state concentration with respect to a point traveling at the mean velocity, i.e. $(\frac{\partial c}{\partial t})_{\xi} = 0$. In the sections that follow it will be shown that these conditions are necessary and sufficient not only for unidirectional flows but for oscillatory flows as well provided that steady state is defined in the periodic sense. It will also be shown that Taylor's assumptions regarding the functional dependence of $\bar{c}$ and c'' on ξ , y , and z as stated in I.B.1 follow directly from the more general requirements of steady state u and c , and uniform flow.

II-B Unidirectional Flow

To obtain an expression for E_L as defined by (II-13) it is necessary first to have solutions for u'' and c'' . Ideally a solution for $u''(y,z)$ is obtained from the governing equation of motion in the x direction. This solution is then used to force a simplified form of (II-8) to obtain a solution for c'' which is then correlated with u'' according to (II-13) for the determination of the longitudinal dispersion coefficient. As stated in I.B.1 Taylor reduced an expression similar to (II-8) to a solvable form by assuming:

- (1) c'' is not a function of ξ .
- (2) $\frac{\partial \bar{c}}{\partial \xi}$ is constant.
- (3) Steady state conditions with respect to the moving coordinate system exist, i.e. $(\frac{\partial c}{\partial t})_{\xi} = 0$.

Aris, however, showed that assumption (3) along with the requirements of steady uniform flow were necessary and sufficient for the definition of a dispersion coefficient as stated by (II-13), and it will now be demonstrated that assumptions (1) and (2) follow from these conditions.

For steady state conditions to exist (II-11) reduces to

$$0 = - \frac{\partial}{\partial \xi} \frac{1}{A} \iint_A u'' c'' dA$$

Assuming the channel to be prismatic, the order of differentiation and integration may be reversed yielding

$$0 = \iint_A \frac{\partial}{\partial \xi} (u'' c'') dA$$

which for uniform flow becomes

$$0 = \iint_A u'' \frac{\partial c''}{\partial \xi} dA \quad (\text{II-14})$$

To examine the functional dependence of c'' , let

$$u'' = g_2(y, z)$$

$$\frac{\partial c''}{\partial \xi} = f_2(y, z, \xi)$$

Equation (II-14) may then be written as

$$0 = \iint_A g_2(y, z) \cdot f_2(y, z, \xi) dA \quad (\text{II-15})$$

which upon integration yields

$$G(\xi) = 0 \quad (\text{II-16})$$

From (II-16) it may be inferred that $\frac{\partial c''}{\partial \xi} = 0$ or c'' is not a function of ξ thereby justifying Taylor's first assumption restated as

$$c(x, y, z) = \bar{c}(x) + c''(y, z) \quad (\text{II-17})$$

The above result is then used to substantiate Taylor's second assumption that $\frac{\partial \bar{c}}{\partial \xi}$ is constant. To do this (II-8) is differentiated with respect to ξ . For steady state conditions and uniform flow the result is

$$u'' \frac{\partial^2 \bar{c}}{\partial \xi^2} = 0 \quad (\text{II-18})$$

which upon integration yields

$$\frac{\partial \bar{c}}{\partial \xi} = \text{constant} \quad (\text{II-19})$$

The application of (II-17) and (II-19) reduces (II-8) to

$$\bar{K}_y \frac{\partial^2 c''}{\partial y^2} + \bar{K}_z \frac{\partial^2 c''}{\partial z^2} = u'' \frac{\partial \bar{c}}{\partial \xi} \quad (\text{II-20})$$

where $\frac{\partial \bar{c}}{\partial \xi}$ is constant.

Thus it is shown that the definition of the longitudinal dispersion coefficient as stated by (II-13), which implies the use of solutions for u'' and c'' obtained from the governing equation of motion and (II-20) respectively, is valid for free surface flows provided the necessary and sufficient conditions of uniform steady flow and steady state concentration distributions with respect to a coordinate system traveling with the mean velocity are satisfied. Taylor's assumptions as stated by (II-17) and (II-19) follow from these conditions.

II-C Oscillatory Flow

The same analysis carried out in II-B for steady unidirectional flow can be extended to the oscillatory flow case if it is assumed that steady state conditions in the oscillatory sense can be satisfied by requiring both the velocity, u , and the concentration, c , to be temporally periodic. This, of course, implies that $\bar{u}$, $\bar{c}$, u'' , and c'' are also temporally periodic.

To show that Taylor's same assumptions apply to dispersion in oscillatory flows it is again necessary only to require that the flow be uniform and that the velocity and concentration be steady state functions in the oscillatory sense, i.e., periodic. If this can be shown to be true then it is reasonable to assume that Aris' argument could be extended to periodic functions thereby justifying the definition of a longitudinal dispersion coefficient in oscillatory flow using Taylor's method.

The analysis is begun by averaging (II-11) over one full period of oscillation which for a periodic c yields

$$\left\langle -\frac{\partial}{\partial \xi} (\bar{u''c''}) \right\rangle_T = 0 \quad (\text{II-21})$$

where ξ is now defined by

$$\xi = x - \int_0^t \bar{u}(t') dt' \quad (\text{II-22})$$

Next, it is assumed once again that the time rate of change in depth is small so that the implied order of integration and differentiation in (II-21) may be reversed

$$\frac{\partial}{\partial \xi} \int_0^h \int_{-w/2}^{w/2} \int_0^T u''c'' dt dy dz = 0 \quad (\text{II-23})$$

Since it is required that the flow be uniform and that u and c be

temporally periodic, assume

$$\begin{aligned}u'' &= g_1(y, z) \cos \sigma t \\c'' &= f_1(y, z, \xi) \cos(\sigma t - \alpha_1)\end{aligned}$$

where α_1 is a constant. It then follows that the temporal integration of $u''c''$ is of the form

$$\int_0^T u''c'' dt = B(y, z, \xi) \cos \alpha_1 \quad (\text{II-24})$$

Substituting (II-24) into (II-23)

$$\frac{\partial}{\partial \xi} \int_0^h \int_{-w/2}^{w/2} B(y, z, \xi) \cos \alpha_1 dy dz = 0$$

which upon integration reduces to the form

$$\cos \alpha_1 \cdot \frac{\partial \chi(\xi)}{\partial \xi} = 0 \quad (\text{II-25})$$

From this it is seen that $\chi(\xi)$ is a constant and thus c'' is not a function of ξ . This is the same result obtained in II-B and is consistent with Taylor's first assumption. However, for the oscillatory case there is an additional distinction to be made. Recognizing that uniform flow is required and c'' is not a function of ξ , (II-11) reduces to

$$\left(\frac{\partial \bar{c}}{\partial t}\right)_{\xi} = 0 \quad (\text{II-26})$$

and (II-8) becomes

$$\left(\frac{\partial c''}{\partial t}\right)_{\xi} - \bar{K}_y \frac{\partial^2 c''}{\partial y^2} - \bar{K}_z \frac{\partial^2 c''}{\partial z^2} = -u'' \frac{\partial \bar{c}}{\partial \xi} \quad (\text{II-27})$$

Differentiating (II-27) with respect to ξ yields

$$u'' \frac{\partial^2 \bar{c}}{\partial \xi^2} = 0$$

or, $\frac{\partial \bar{c}}{\partial \xi} = \text{constant}$, which is again consistent with Taylor's assumption for the unidirectional flow case. Equation (II-27) with $\frac{\partial \bar{c}}{\partial \xi}$ constant is the comparable form of (II-20) for oscillatory flow. It is from the solution of this equation that $c''(y,z,t)$ is obtained for determination of the longitudinal dispersion coefficient, E_x .

Based upon the above discussion it is concluded that Taylor's analysis, under the conditions of uniform flow and temporal periodicity of u and c , is capable of being extended for the determination of longitudinal dispersion coefficients in oscillatory flows and remains consistent with the conditions for which a dispersion coefficient may be defined as shown by Aris.

II-D Application of Dispersion Coefficients Predicted by Taylor's Method

It is not the purpose of the present work to provide a detailed analysis of the conditions under which values of E_L and E_x predicted by Taylor's method may be applied for the determination of concentration distributions. However, a comment regarding this matter is considered useful here in order to retain a proper perspective on the work presented.

Equation (II-12) indicates that in applying values of the dispersion coefficient, steady state conditions for the cross-sectional mean concentration are not satisfied whereas in Taylor's analysis for the determination of the dispersion coefficient it is assumed that these steady state conditions exist. Aris (3) has shown that the steady state conditions required for the convective mass transport to behave as a Fickian flux are approached asymptotically at a rate determined by the ratio of the time of convective flow to the time of turbulent cross-sectional mixing. This interpretation was also given by Taylor (1) on a qualitative basis. Fischer (21) then reasoned that a necessary condition for the application of coefficients obtained in this manner for the prediction of concentration is that c'' be everywhere much less than $\bar{c}$. This in effect strikes a compromise between the strict steady state requirement for the definition of a dispersion coefficient and the conditions for which it may be used. Fischer (21) among others has developed detailed criteria for the application of dispersion coefficients in (II-12).

CHAPTER III
DISPERSIVE MASS TRANSPORT IN AN INFINITELY
WIDE RECTANGULAR CHANNEL

III-A Steady Unidirectional Flow

The coordinate system has been selected (Figure 2) with origin at the bed of the channel, z coordinate positive upwards, and x coordinate positive in the direction of flow, u .

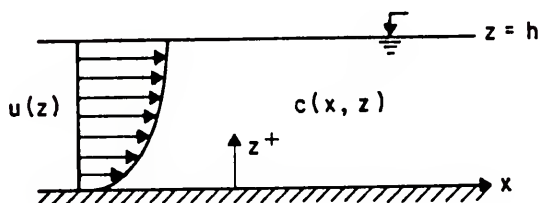


FIGURE 2. DEFINITION SKETCH FOR TWO
DIMENSIONAL SHEAR FLOW

The general procedures used throughout Chapters III and IV will be to first obtain analytical solutions for u'' and c'' and then use these to calculate the dispersive mass transport $\overline{u''c''}$ from which the dispersion coefficient is determined.

III-A.1 Velocity Distribution

It is assumed that the turbulent shear stress may be expressed in terms of the Boussinesq approximation and that the eddy viscosity, ϵ_z , is constant and equal to its mean value over depth. The equation of motion for two dimensional steady uniform shear flow is then

$$0 = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \epsilon_z \frac{\partial^2 u}{\partial z^2} \quad (\text{III-1})$$

where ρ is the density of the fluid and P is pressure. It will be

assumed in all analyses that turbulent fluctuations have been averaged and incorporated into the remaining terms. The pressure gradient term for this case may then be expressed as

$$\frac{1}{\rho} \frac{\partial P}{\partial x} = K \quad (\text{III-2})$$

where K is constant. Substituting (III-2) into (III-1) and requiring the boundary conditions of no slip at the bed and zero shear stress at the surface the flow boundary value problem is then

$$\text{DE: } \frac{d^2 u}{dz^2} = \frac{K}{\bar{\epsilon}_z} \quad (\text{III-3})$$

$$\text{BC's: } u(0) = 0$$

$$\left. \frac{du}{dz} \right|_{z=h} = 0$$

The velocity shear profile is obtained by integration of (III-3) and the application of the boundary conditions. The resulting expression is

$$u(z) = \frac{K}{\bar{\epsilon}_z} \left(\frac{z^2}{2} - hz \right) \quad (\text{III-4})$$

The spatial variation of the velocity as defined by (II-2) is then obtained by averaging (III-4) over depth and subtracting out the depth mean component to yield

$$u''(z) = \frac{K}{\bar{\epsilon}_z} \left(\frac{z^2}{2} - hz + \frac{h^2}{3} \right) \quad (\text{III-5})$$

The portion of the unidirectional flow analysis remaining in this section and portions of the analyses in later sections treat the expression of the variations of velocity and concentration and the dispersion coefficient as functions of normalizing parameters other than the pressure gradient modulus, K . In some cases these alternate formulations are considered to be more useful for application purposes, whereas in other cases they are presented for the comparison of longitudinal dispersive

mass transport in oscillatory and unidirectional flow or in oscillatory flows of differing periods of oscillation. Therefore, before proceeding further a brief discussion follows describing these alternate formulations and the rationale used in their selection.

As noted in Chapter I the approach taken by most investigators in describing dispersive mass transport in oscillatory flows has been to first assume a velocity profile that is at each instant in time similar in form to a corresponding unidirectional flow, and then use this to obtain an expression for the dispersion coefficient, E_x , either directly through the method of moments or indirectly through Taylor's method of describing the convective mass transport $\overline{u''c''}$ as a Fickian flux. The dispersion coefficients thus obtained have then been analyzed only as they relate to a "corresponding" unidirectional flow coefficient. The approach taken here is that it is not realistic to assume a priori that the dispersive mechanism in oscillatory flow is functionally similar to the same mechanism in unidirectional flow because of the distinct differences in the flow characteristics of the two cases. This argument may also be applied to the more general case of comparing dispersive mass transport characteristics of oscillatory flows of different periods. It will later be shown that such comparisons are easily misleading and if done at all they must be done with a full understanding of the mechanism involved. To illustrate this point, there are many ways in which unidirectional and oscillatory, or two oscillatory mass transport cases of different periods, can be considered to be "corresponding." Normalizing parameters considered in the present analysis for comparison of "corresponding" dispersion coefficients are:

- (1) pressure gradient modulus, K ,
- (2) maximum cross-sectional velocity, $u_{\max}$, which for the most general case of three dimensional oscillatory shear flow would represent the amplitude of the periodic velocity function located on the centerline of the channel at the surface,
- (3) surface excursion length, L , experienced by a particle on the free surface over a period of time equal to one-half the period of oscillation.

For the case of a unidirectional flow, such as that being considered in this section, the concept of an excursion length as described above has little physical meaning. Thus, excursion length normalization of u'' , c'' , and E_x solutions will be made for oscillatory flows only. The expression of (III-5) in terms of $u_{\max}$ is easily accomplished by solving (III-4) for K at $z = h$, or

$$K = - \frac{2\bar{c}_z u_{\max}}{h} \quad (\text{III-6})$$

which upon substitution into (III-5) yields

$$u''(z) = - \frac{2u_{\max}}{h^2} \left(\frac{z^2}{2} - hz + \frac{h^2}{3} \right) \quad (\text{III-7})$$

III-A.2 Concentration Distribution

A solution for $c''(z)$ as a function of K is obtained by using the solution for the spatial variation of velocity, $u''(z)$, given by (III-5), to force the one dimensional form of (II-20). Thus, the concentration variation boundary value problem may be stated as

$$\underline{\text{DE:}} \quad \bar{K}_z \frac{d^2 c''}{dz^2} = u'' \frac{\partial \bar{c}}{\partial \xi} \quad (\text{III-8})$$

$$\underline{\text{BC's:}} \quad z = 0, h; \quad \frac{dc''}{dz} = 0$$

where,

$$\frac{\partial \bar{c}}{\partial \xi} = \text{constant}$$

$u''(z)$ is given by (III-5).

Substituting for u'' , integrating once with respect to z , and applying the boundary condition at $z = 0$ produces

$$\frac{dc''}{dz} = \frac{K}{\bar{\epsilon}_z \bar{K}_z} \cdot \left(\frac{\partial \bar{c}}{\partial \xi} \right) \left[\frac{z^3}{6} - \frac{hz^2}{2} + \frac{h^2z}{3} \right] \quad (\text{III-9})$$

It should be noted that (III-9) implicitly satisfies the boundary condition at $z = h$ since by definition $\int_0^h u''(z) dz = 0$. Equation (III-9) is then integrated once more over z and the constant of integration is evaluated using the requirement that $c''(z)$ must have a zero mean over depth. The resulting solution for the concentration variation is then

$$c''(z) = \frac{K}{\bar{\epsilon}_z \bar{K}_z} \left(\frac{\partial \bar{c}}{\partial \xi} \right) \left[\frac{z^4}{24} - \frac{hz^3}{6} + \frac{h^2z^2}{6} - \frac{h^4}{45} \right] \quad (\text{III-10})$$

Using (III-6) to substitute for K in (III-10) the expression for $c''(z)$ as a function of $u_{\max}$ is

$$c''(z) = - \frac{2u_{\max}}{K_z h^2} \left(\frac{\partial \bar{c}}{\partial \xi} \right) \left[\frac{z^4}{24} - \frac{hz^3}{6} + \frac{h^2z^2}{6} - \frac{h^4}{45} \right] \quad (\text{III-11})$$

III-A.3 Dispersive Mass Transport

The dispersive mass transport over the depth of flow is given by

$$\dot{m} = \int_0^h u'' c'' dz \quad (\text{III-12})$$

Substituting for u'' and c'' using (III-5) and (III-10) respectively, (III-12) becomes

$$\dot{m} = \frac{K^2}{\bar{\epsilon}_z \bar{K}_z} \left(\frac{\partial \bar{C}}{\partial \bar{\xi}} \right) \int_0^h \left(\frac{z^2}{2} - hz + \frac{h^2}{3} \right) \left(\frac{z^4}{24} - \frac{hz^3}{6} + \frac{h^2 z^2}{6} - \frac{h^4}{45} \right) dz$$

Carrying out the integration the mass transport is then

$$\dot{m} = \left(- \frac{\partial \bar{C}}{\partial \bar{\xi}} \right) \cdot \frac{2K^2 h^7}{945 \bar{\epsilon}_z^2 \bar{K}_z} \quad (\text{III-13})$$

It shall be assumed for this analysis and for the analyses that follow that Reynolds analogy relating the transfer of mass and momentum by turbulent processes is applicable so that

$$\bar{\epsilon}_z = \bar{K}_z \quad (\text{III-14})$$

This assumption is not necessary to obtain solutions for E_L and E_x using the method presented here; however as will be seen later its use simplifies the forms of solutions considerably and facilitates interpretation of the physical processes involved. Thus, with the aid of (III-14) and the introduction of the vertical mixing time defined by

$$T_{cz} = \frac{h^2}{\bar{K}_z} \quad (\text{III-15})$$

Equation (III-13) may be written as

$$\dot{m} = \left(- \frac{\partial \bar{C}}{\partial \bar{\xi}} \right) \frac{2K^2 T_{cz}^3 h}{945} \quad (\text{III-16})$$

Assuming next that the required conditions for defining a longitudinal dispersion coefficient as discussed in II-B are satisfied, then

$$E_L = \frac{\dot{m}}{\left(- \frac{\partial \bar{C}}{\partial \bar{\xi}} \right) h}$$

or,

$$E_L = \frac{2K^2 T^3}{945} \frac{cz}{cz} \quad (\text{III-17})$$

Using (III-6) and (III-15) in (III-17) the longitudinal dispersion coefficient as a function of $u_{\max}$ is

$$E_L = \frac{8 \cdot u_{\max}^2 T}{945} \frac{cz}{cz} \quad (\text{III-18})$$

The functional form of E_L given by (III-18) is the same as that obtained by Taylor for laminar unidirectional flow in a tube, (I-8). In both cases the longitudinal dispersion varies directly as the product of the square of the velocity and the cross-sectional mixing time. Thus, for a given unidirectional velocity profile the longitudinal mass transport of a substance increases linearly with the time required to mix that substance over the flow cross section by diffusive processes.

III-B Oscillatory Flow

III-B.1 Velocity Distribution

The same assumptions used in III-A regarding selection of coordinate system, and the use of the Boussinesq approximation for expressing the viscous term in the equation of motion will be applied here. The governing equation of fluid motion for uniform unsteady two dimensional shear flow is then

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \bar{\epsilon}_z \frac{\partial^2 u}{\partial z^2} \quad (\text{III-19})$$

For the oscillatory flow case it will be assumed that long wave phenomena are of primary interest and therefore the pressure distribution over depth is hydrostatic. It is then reasonable to assume that the pressure gradient in (III-19) is periodic and of the form

$$\frac{1}{\rho} \frac{\partial P}{\partial x} = K e^{i\sigma t} \quad (\text{III-20})$$

where K is constant, as before, and σ is the angular frequency of oscillation. Note that only the real part of (III-20) has physical meaning.

To obtain a solution assume that $u(z,t)$ is periodic in time and of the form

$$u(z,t) = u_s(z) e^{i\sigma t} \quad (\text{III-21})$$

where $u_s(z)$ is a complex function. The boundary value problem is then formulated by incorporating (III-20) and (III-21) into (III-19) which along with the boundary conditions of no slip at the channel bed and zero shear stress at the surface becomes

$$\text{DE: } \bar{\epsilon}_z \frac{d^2 u_s}{dz^2} - i\sigma u_s = K \quad (\text{III-22})$$

$$\text{BC's: } u_s(0) = 0$$

$$\left. \frac{du_s}{dz} \right|_{z=h} = 0$$

This problem has been previously solved by Lamb (18) and Segall (19) who obtained

$$u(z,t) = \frac{iK}{\sigma} \left[1 - \frac{\cosh \beta(1+i)(h-z)}{\cosh \beta(1+i)h} \right] e^{i\sigma t} \quad (\text{III-23})$$

where,

$$\beta = \sqrt{\frac{\sigma}{2\varepsilon_z}} \quad (\text{III-24})$$

with only the real part of (III-23) having any physical meaning. Its form is clearly that of a damped progressive shear wave propagating upward through the water column and causing a temporal phase shift in the velocity as a function of z . It was also pointed out by Segall and Gidlund (20) that the solution as expressed by (III-23) correctly predicts flow reversals in the lower momentum layers of the fluid near the bed prior to a shift in flow direction of the higher momentum layers near the free surface. These effects are extremely important to the dispersive transport mechanism.

The spatial variation of the velocity, $u''(z,t)$, is obtained by averaging (III-23) over depth and subtracting out the mean from the total velocity according to

$$u''(z,t) = u(z,t) - \bar{u}(t)$$

The result is then

$$u''(z,t) = \frac{iKe^{i\sigma t}}{\sigma \cosh \beta(1+i)h} \left\{ \frac{\sinh \beta(1+i)h}{\beta(1+i)h} - \cosh \beta(1+i)(h-z) \right\} \quad (\text{III-25})$$

For reasons that will become obvious in the next section the velocity variation solution as given by (III-25) will now be expanded in a Fourier cosine series of the form

$$u''(z,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi z}{h} e^{i\sigma t} \quad (\text{III-26})$$

where coefficients a_n are complex. The a_0 term in the series has been omitted to satisfy the zero mean requirement for u'' . It is noted by Hildebrand (22) that any piecewise differentiable function may be completely represented by a cosine series of the form $a_n \cos \frac{n\pi z}{h}$ over the closed interval $0 \leq z \leq h$. Proceeding then,

$$a_n = \frac{2}{h} \int_0^h \frac{iK \sinh \beta(1+i)h}{\sigma \beta(1+i)h \cosh \beta(1+i)h} \cos \frac{n\pi z}{h} dz \\ - \frac{2}{h} \int_0^h \frac{iK}{\sigma \cosh \beta(1+i)h} \cdot \cosh \beta(1+i)(h-z) \cos \frac{n\pi z}{h} dz$$

The first term on the right hand side integrates to zero while the second term must be integrated by parts twice which yields for a_n ,

$$a_n = \frac{2K \beta h(1-i) \sinh \beta(1+i)h}{\sigma \cosh \beta(1+i)h [(n\pi)^2 + i2(\beta h)^2]} \quad (\text{III-27})$$

Since only the real part of $u''(z,t)$ is of any interest, (III-27) will now be put in polar form for incorporation into (III-26). Considering each complex term separately,

$$(1-i) = \sqrt{2} e^{-i\frac{\pi}{4}} \quad (\text{III-28})$$

$$\sinh \beta(1+i)h = \frac{\sqrt{2}}{2} (\cosh 2\beta h - \cos 2\beta h)^{\frac{1}{2}} e^{i\alpha_1} \quad (\text{III-29})$$

$$\cosh \beta(1+i)h = \frac{\sqrt{2}}{2} (\cosh 2\beta h + \cos 2\beta h)^{\frac{1}{2}} e^{i\alpha_2} \quad (\text{III-30})$$

$$[(n\pi)^2 + i2(\beta h)^2] = [(n\pi)^4 + 4(\beta h)^4]^{\frac{1}{2}} e^{i\alpha_3} \quad (\text{III-31})$$

where,

$$\alpha_1 = \tan^{-1} \left(\frac{\cosh \beta h \sin \beta h}{\sinh \beta h \cos \beta h} \right) \quad (\text{III-32})$$

$$\alpha_2 = \tan^{-1} \left(\frac{\sinh \beta h \sin \beta h}{\cosh \beta h \cos \beta h} \right) \quad (\text{III-33})$$

$$\alpha_3 = \tan^{-1} \left[\frac{2(\beta h)^2}{(n\pi)^2} \right] \quad (\text{III-34})$$

To express $u''(z,t)$ in its final form it is again assumed that Reynolds analogy is applicable allowing the use of (III-15). A non-dimensional time relating the period of oscillation to vertical mixing time is then introduced by

$$T'_Z = \frac{T}{T} \frac{cz}{T} \quad (\text{III-35})$$

With the aid of Reynolds analogy and (III-15), (III-24), and (III-35) the βh arguments appearing throughout a_n become

$$\beta h = \sqrt{\pi T'_Z} \quad (\text{III-36})$$

The final form for $u''(z,t)$ as a function of the pressure gradient is arrived at by combining (III-26) through (III-36) to yield,

$$u''(z,t) = KT \left\{ \frac{2T'_Z (\cosh 2\sqrt{\pi T'_Z} - \cos 2\sqrt{\pi T'_Z})}{\pi (\cosh 2\sqrt{\pi T'_Z} + \cos 2\sqrt{\pi T'_Z})} \right\}^{1/2} \cdot \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi z}{h} e^{i(\sigma t - \frac{\pi}{4} + \alpha_1 - \alpha_2 - \alpha_3(n))}}{[(n\pi)^4 + (2\pi T'_Z)^2]^{1/2}} \quad (\text{III-37})$$

where,

$$\alpha_1 = \tan^{-1} \left\{ \frac{\cosh \sqrt{\pi T'_Z} \sin \sqrt{\pi T'_Z}}{\sinh \sqrt{\pi T'_Z} \cos \sqrt{\pi T'_Z}} \right\} \quad (\text{III-38})$$

$$\alpha_2 = \tan^{-1} \left\{ \frac{\sinh \sqrt{\pi T'_Z} \sin \sqrt{\pi T'_Z}}{\cosh \sqrt{\pi T'_Z} \cos \sqrt{\pi T'_Z}} \right\} \quad (\text{III-39})$$

$$\alpha_3(n) = \tan^{-1} \left\{ \frac{2\pi T'_Z}{(n\pi)^2} \right\} \quad (\text{III-40})$$

Expressions for $u''(z,t)$ as a function of the surface amplitude, $u_{\max}$, of the periodic velocity function and the surface excursion length, L , experienced by a particle during one-half of the period of oscillation are obtained by solving for K as a function of $u_{\max}$ and L using

(III-23). Details of this development are presented in Appendix A, the results of which are

$$K = - \frac{\sqrt{2} \pi (\cosh 2 \sqrt{\pi T'_Z} + \cos 2 \sqrt{\pi T'_Z})^{\frac{1}{2}}}{T (\cosh \sqrt{\pi T'_Z} - \cos \sqrt{\pi T'_Z})} u_{\max} \quad (\text{III-41})$$

and

$$K = - \frac{\sqrt{2} \pi^2 L (\cosh 2 \sqrt{\pi T'_Z} + \cos 2 \sqrt{\pi T'_Z})^{\frac{1}{2}}}{T^2 (\cosh \sqrt{\pi T'_Z} - \cos \sqrt{\pi T'_Z})} \quad (\text{III-42})$$

Substitution for K in (III-37) using these two expressions then yields

$$u''(z,t) = - \frac{2u_{\max} \{\pi T'_Z (\cosh 2 \sqrt{\pi T'_Z} - \cos 2 \sqrt{\pi T'_Z})\}^{\frac{1}{2}}}{(\cosh \sqrt{\pi T'_Z} - \cos \sqrt{\pi T'_Z})} \cdot \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi z}{h} e^{i(\sigma t - \frac{\pi}{4} + \alpha_1 - \alpha_2 - \alpha_3)}}{[(n\pi)^4 + (2\pi T'_Z)^2]^{\frac{1}{2}}} \quad (\text{III-43})$$

and,

$$u''(z,t) = - \frac{2\pi L \{\pi T'_Z (\cosh 2 \sqrt{\pi T'_Z} - \cos 2 \sqrt{\pi T'_Z})\}^{\frac{1}{2}}}{T (\cosh \sqrt{\pi T'_Z} - \cos \sqrt{\pi T'_Z})} \cdot \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi z}{h} e^{i(\sigma t - \frac{\pi}{4} + \alpha_1 - \alpha_2 - \alpha_3)}}{[(n\pi)^4 + (2\pi T'_Z)^2]^{\frac{1}{2}}} \quad (\text{III-44})$$

III-B.2 Concentration Distribution

For temporally periodic u and c , uniform flow, and constant eddy diffusivity it was shown in II-C that the transport diffusion equation as described from a coordinate system moving with the cross-sectional mean velocity could be written as

$$\frac{\partial c''}{\partial t} - \bar{K}_z \frac{\partial^2 c''}{\partial z^2} = -u'' \frac{\partial \bar{c}}{\partial \xi} \quad (\text{III-45})$$

where the rate of change with respect to ξ will hereafter be implied.

The concentration variation, $c''(z,t)$, shall be assumed to be of the form

$$c''(z,t) = c''_S(z)e^{i\sigma t} = \sum_{m=1}^{\infty} a'_m \cos \frac{m\pi z}{h} e^{i\sigma t} \quad (\text{III-46})$$

where a'_m is complex. This expression for c'' is seen to satisfy the requirements of periodicity and zero diffusive flux of substance across the flow boundaries. Moreover, if (III-46) is substituted into (III-45) and operated on, the left hand side is composed of $\cos \frac{n\pi z}{h}$ terms only. Thus, to satisfy the conditions of equality, the expansion of (III-25) for u'' in a $\cos \frac{n\pi z}{h}$ series is justified. The formal statement of the boundary value problem for $c''_S(z)$ is then

$$\text{DE: } \bar{K}_z \frac{d^2 c''_S}{dz^2} - i\sigma c''_S = u''_S \frac{\partial \bar{c}}{\partial \xi} \quad (\text{III-47})$$

$$\text{BC's: } \frac{dc''}{dz} \Big|_{z=0,h} = 0$$

where $\frac{\partial \bar{c}}{\partial \xi}$ is constant and $u''_S(z)$ is defined by (III-26). Substitution for c''_S and u''_S transforms (III-47) to

$$\sum_{m=1}^{\infty} \{ \bar{K}_z \left(\frac{m\pi}{h} \right)^2 + i\sigma \} a'_m \cos \frac{m\pi z}{h} = \left(- \frac{\partial \bar{c}}{\partial \xi} \right) \cdot \sum_{n=1}^{\infty} a_n \cos \frac{n\pi z}{h}$$

which due to the orthogonal properties of $\cos \frac{m\pi z}{h}$ and $\cos \frac{n\pi z}{h}$ requires that $m=n$ for a non-trivial solution. The Fourier coefficient a'_m can then be written as

$$a'_m = \left(- \frac{\partial \bar{c}}{\partial \xi} \right) \frac{h^2 a_m}{\bar{K}_z \left[(m\pi)^2 + i \left(\frac{\sigma h^2}{\bar{K}_z} \right) \right]} \quad (\text{III-48})$$

The required solution for $c''(z,t)$ is arrived at by: (1) using (III-26) and (III-37) to determine the polar form of a'_m ; (2) transforming

the denominator of (III-48) to polar form; (3) applying the definitions for T_{cz} and T'_z where necessary; and (4) substituting the resulting expression into (III-46). The variation component of concentration as a function of the pressure gradient is then,

$$c''(z,t) = \left(-\frac{\partial \bar{C}}{\partial \xi}\right) K T^2 T_z'^{\frac{3}{2}} \left\{ \frac{2(\cosh 2\sqrt{\pi T_z'} - \cos 2\sqrt{\pi T_z'})}{\pi(\cosh 2\sqrt{\pi T_z'} + \cos 2\sqrt{\pi T_z'})} \right\}^{\frac{1}{2}} \cdot$$

$$\sum_{m=1}^{\infty} \frac{\cos \frac{m\pi z}{h} e^{i(\sigma t - \frac{\pi}{4} + \alpha_1 - \alpha_2 - 2\alpha_3(m))}}{[(m\pi)^4 + (2\pi T_z')^2]} \quad (III-49)$$

where α_1 , α_2 , and $\alpha_3(m)$ are given by (III-38) through (III-40) with m replacing n in (III-40).

Solutions for $c''(z,t)$ as a function of u_{max} , and L are obtained by applying (III-41) and (III-42) to (III-49), to yield

$$c''(z,t) = 2u_{max} T_z'^{\frac{3}{2}} T \left(\frac{\partial \bar{C}}{\partial \xi}\right) \frac{[\pi(\cosh 2\sqrt{\pi T_z'} - \cos 2\sqrt{\pi T_z'})]^{\frac{1}{2}}}{(\cosh \sqrt{\pi T_z'} - \cos \sqrt{\pi T_z'})} \cdot$$

$$\sum_{m=1}^{\infty} \frac{\cos \frac{m\pi z}{h} e^{i(\sigma t - \frac{\pi}{4} + \alpha_1 - \alpha_2 - 2\alpha_3(m))}}{[(m\pi)^4 + (2\pi T_z')^2]} \quad (III-50)$$

and

$$c''(z,t) = 2L(\pi T_z')^{\frac{3}{2}} \left(\frac{\partial \bar{C}}{\partial \xi}\right) \frac{(\cosh 2\sqrt{\pi T_z'} - \cos 2\sqrt{\pi T_z'})^{\frac{1}{2}}}{(\cosh \sqrt{\pi T_z'} - \cos \sqrt{\pi T_z'})} \cdot$$

$$\sum_{m=1}^{\infty} \frac{\cos \frac{m\pi z}{h} e^{i(\sigma t - \frac{\pi}{4} + \alpha_1 - \alpha_2 - 2\alpha_3(m))}}{[(m\pi)^4 + (2\pi T_z')^2]} \quad (III-51)$$

III-B.3 Dispersive Mass Transport

For the oscillatory flow case the mass transport considered will be the time mean transport averaged over one period of oscillation. Thus,

the dispersive mass transport is given by

$$\langle \dot{m} \rangle_T = \left\langle \int_0^h R_e(u'') \cdot R_e(c'') dz \right\rangle_T \quad (\text{III-52})$$

Using (III-37) and (III-49) for $u''(z,t)$ and $c''(z,t)$, the integration over depth of the product $R_e(u'') \cdot R_e(c'')$ reduces the implied double summation over m and n to a single sum through the orthogonal properties of the functions $\cos \frac{n\pi z}{h}$ and $\cos \frac{m\pi z}{h}$ since,

$$\int_0^h \cos \frac{n\pi z}{h} \cos \frac{m\pi z}{h} dz = \begin{cases} \frac{h}{2}, & n = m \\ 0, & n \neq m \end{cases}$$

Performing this integration and averaging the results over T yields,

$$\langle \dot{m} \rangle_T = \left(-\frac{\partial \bar{c}}{\partial \xi} \right) \frac{h K^2 T^3 T_z'^2}{2\pi} \cdot \frac{(\cosh 2\sqrt{\pi T_z'} - \cos 2\sqrt{\pi T_z'})}{(\cosh 2\sqrt{\pi T_z'} + \cos 2\sqrt{\pi T_z'})} \cdot \sum_{n=1}^{\infty} \frac{\cos \alpha_3(n)}{[(n\pi)^4 + (2\pi T_z')^2]^{\frac{3}{2}}} \quad (\text{III-53})$$

The longitudinal dispersion coefficient is next introduced using Taylor's analogy defining the convective mass transport $\langle \dot{m} \rangle_T$ as a Fickian flux so that

$$\langle \dot{m} \rangle_T = -E_x h \left(\frac{\partial \bar{c}}{\partial \xi} \right)_T \quad (\text{III-54})$$

Noting from (III-40) that

$$\cos \alpha_3(n) = \frac{(n\pi)^2}{[(n\pi)^4 + (2\pi T_z')^2]^{\frac{3}{2}}}$$

and combining (III-53) and (III-54) the time mean longitudinal dispersion coefficient as a function of the pressure gradient is expressed as

$$\langle E_x \rangle_T = \frac{K^2 T^3 T_z'^2}{2\pi} \cdot \frac{(\cosh 2 \sqrt{\pi T_z'} - \cos 2 \sqrt{\pi T_z'})}{(\cosh 2 \sqrt{\pi T_z'} + \cos 2 \sqrt{\pi T_z'})} \cdot \sum_{n=1}^{\infty} \frac{(n\pi)^2}{[(n\pi)^4 + (2\pi T_z')^2]^2} \quad (\text{III-55})$$

Solutions for $\langle E_x \rangle_T$ as a function of $u_{\max}$ and L may be obtained by substituting directly for K^2 in (III-55) using (III-41) and (III-42).

The resulting expressions are as follows:

$$\langle E_x \rangle_T = \pi u_{\max}^2 T T_z'^2 \frac{(\cosh 2 \sqrt{\pi T_z'} - \cos 2 \sqrt{\pi T_z'})}{(\cosh \sqrt{\pi T_z'} - \cos \sqrt{\pi T_z'})^2} \cdot \sum_{n=1}^{\infty} \frac{(n\pi)^2}{[(n\pi)^4 + (2\pi T_z')^2]^2} \quad (\text{III-56})$$

and,

$$\langle E_x \rangle_T = \frac{\pi^3 L^2 T_z'^2}{T} \frac{(\cosh 2 \sqrt{\pi T_z'} - \cos 2 \sqrt{\pi T_z'})}{(\cosh \sqrt{\pi T_z'} - \cos \sqrt{\pi T_z'})^2} \cdot \sum_{n=1}^{\infty} \frac{(n\pi)^2}{[(n\pi)^4 + (2\pi T_z')^2]^2} \quad (\text{III-57})$$

Equation (III-56) is considered to be the most useful and descriptive of the solutions presented for $\langle E_x \rangle_T$ in two dimensional oscillatory shear flow. Therefore, to facilitate a discussion of these results in the next section, (III-56) is non-dimensionalized by defining

$$E'_x = \frac{\langle E_x \rangle_T}{u_{\max}^2 T} \quad (\text{III-58})$$

and restating it as

$$E'_x = \frac{\pi T_z'^2 (\cosh 2 \sqrt{\pi T_z'} - \cos 2 \sqrt{\pi T_z'})}{(\cosh \sqrt{\pi T_z'} - \cos \sqrt{\pi T_z'})^2} \cdot$$

$$\sum_{n=1}^{\infty} \frac{(n\pi)^2}{[(n\pi)^4 + (2\pi T_z')^2]^2} \quad (\text{III-59})$$

III-C Discussion of Two Dimensional Shear Flow Results

III-C.1 Dispersion in Unidirectional Flow as a Limit of the Oscillatory Flow Case

This discussion shall begin by relating the oscillatory flow dispersion coefficient, $\langle E_X \rangle_T$, to its "corresponding" unidirectional flow coefficient, E_L , as has been previously done by Holley et al., Awaya, and Fukuoka; and shall then proceed to show how the unidirectional process is in fact a limiting case of an oscillatory process which has its own distinct characteristics. Thus, solutions obtained in III-B.3 for $\langle E_X \rangle_T$ as a function of the pressure gradient and the surface amplitude of the velocity are normalized by the "corresponding" solutions for E_L obtained in III-A.3. Dividing (III-55) by (III-17) and (III-56) by (III-18) yields

$$\frac{\langle E_X \rangle_T}{E_L} = \frac{945}{4\pi T_Z'} \frac{(\cosh 2 \sqrt{\pi T_Z'} - \cos 2 \sqrt{\pi T_Z'})}{(\cosh 2 \sqrt{\pi T_Z'} + \cos 2 \sqrt{\pi T_Z'})} \cdot \sum_{n=1}^{\infty} \frac{(n\pi)^2}{[(n\pi)^4 + (2\pi T_Z')^2]^2} \quad (\text{III-60})$$

for the same K , and

$$\frac{\langle E_X \rangle_T}{E_L} = \frac{945\pi T_Z'}{8} \frac{(\cosh 2 \sqrt{\pi T_Z'} - \cos 2 \sqrt{\pi T_Z'})}{(\cosh \sqrt{\pi T_Z'} - \cos \sqrt{\pi T_Z'})^2} \cdot \sum_{n=1}^{\infty} \frac{(n\pi)^2}{[(n\pi)^4 + (2\pi T_Z')^2]^2} \quad (\text{III-61})$$

for the same $u_{\max}$. All calculations involving infinite summations for two dimensional shear flow dispersion use 200 terms in each sum, well within fourth place accuracy.

The expressions given by (III-60) and (III-61) are plotted in Figure 3 vs. the non-dimensional time $1/T'_Z$ corresponding to T' defined by Holley et al. Also included in Figure 3 is the solution obtained by Holley et al. for $\langle E_X \rangle_T / E_L$ for the oscillatory linear velocity profile discussed in I-B.2. It is noted that all three solutions approach the limiting value of one-half for situations when the period of oscillation is greater than the vertical mixing time in accordance with the findings of Okubo and Bowden. However, the difference in behavior of the two solutions given by (III-60) and (III-61) is dramatic and immediately raises the question of whether or not a plot such as Figure 3 is the most meaningful method of illustrating the characteristics of dispersive mass transport in oscillatory flow. For the case of the pressure gradient normalized solutions the ratio $\langle E_X \rangle_T / E_L$ decreases rapidly with increasing T'_Z because:

- (1) For unidirectional flow the pressure gradient and viscous forces are in equilibrium so as to produce a constant unidirectional shear flow for the dispersive transport of substance.
- (2) In oscillatory flow the pressure gradient is assumed to be simple harmonic in time and therefore is in constant balance with the varying inertia and friction forces. As the period of oscillation decreases the inertial effects become very large such that in the limiting case little or no flow would be induced. This results in little or no dispersive transport. This effect of the period of oscillation on velocity can be seen from (III-23).

The solutions normalized by $u_{\max}$ exhibit a much slower decrease of $\langle E_X \rangle_T / E_L$ with increasing T'_Z simply because in this case the surface

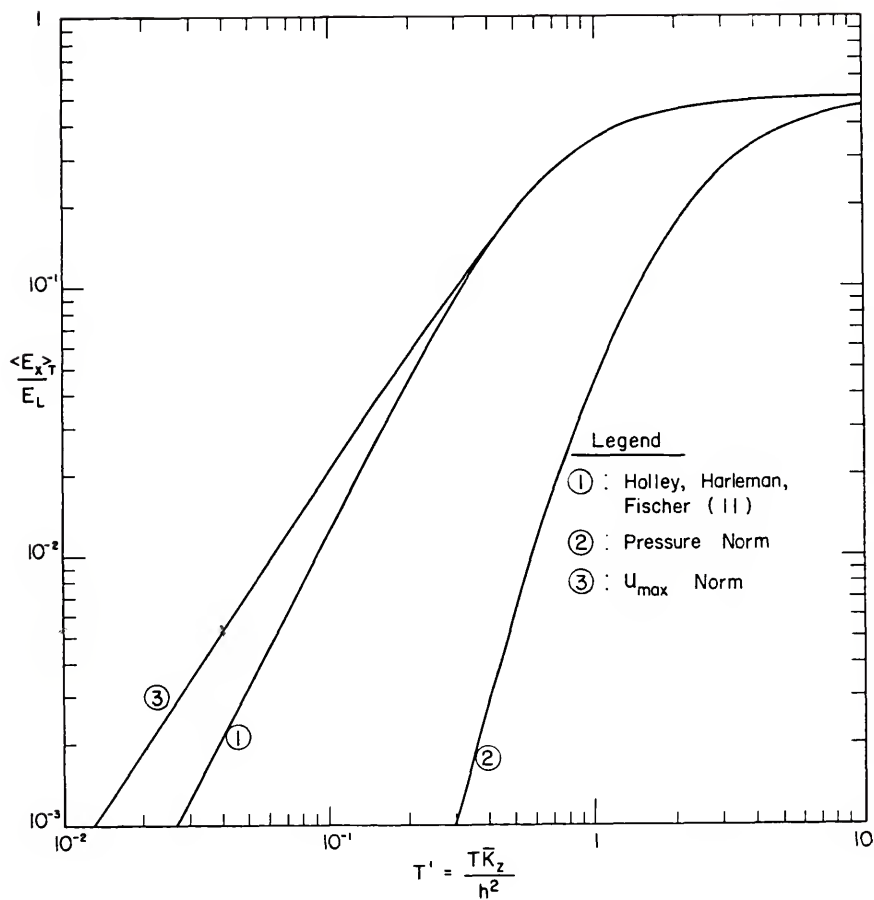


FIGURE 3 NON-DIMENSIONAL DISPERSION COEFFICIENT
 $\langle E_x \rangle_T / E_L$, AS A FUNCTION OF T'

velocities of the unidirectional and oscillatory flow are required to be the same. In terms of pressure gradients this would correspond to the situation where

$$K_{osc} \gg K_{unidir}$$

Based upon these results it becomes apparent that due to the very different hydrodynamic characteristics of unidirectional and oscillatory flow systems, the interpretation of the nature of dispersive mass transport in an oscillatory system as compared to the same process in a unidirectional system is misleading and can vary widely depending upon how one chooses to relate the two systems. This is demonstrated by Figure 4. In this figure the longitudinal dispersion coefficient, $\langle E_x \rangle_T$, as given by (III-56) is plotted against the vertical mixing time, T_{cz} , for four periods of oscillation ranging from 22,536 to 223,560 seconds and a u_{max} of 1 ft/sec. Also plotted is the solution for the unidirectional flow coefficient, E_L , as given by (III-18). As shown by Figure 4 the behavior of E_L and $\langle E_x \rangle_T$ are clearly different. The unidirectional flow coefficient varies directly with the vertical mixing time which for little or no turbulent mixing over the water column allows the shear flow to transport higher concentrations of substance far downstream. The behavior of the oscillatory flow coefficient, however, is governed by a type of resonant interaction between the period of oscillation and the vertical mixing time. As the period of oscillation is increased the resonant peak shifts to the right and likewise increases until the limiting case is reached where the peak is infinitely large and the values predicted for $\langle E_x \rangle_T$ are exactly one-half those predicted for the unidirectional flow coefficient E_L .

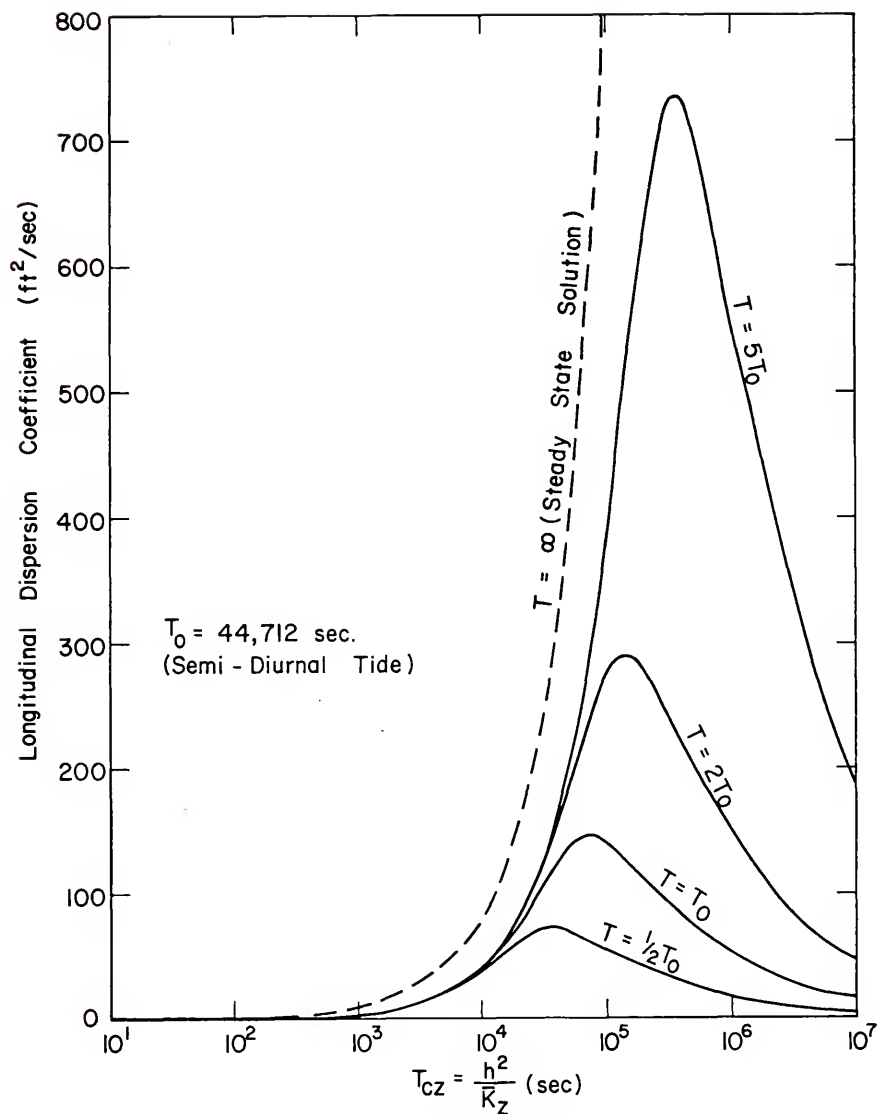


FIGURE 4 DISPERSION IN UNIDIRECTIONAL FLOW AS A LIMIT OF THE OSCILLATORY FLOW CASE

The resonant characteristic of $\langle E_x \rangle_T$ may also be used to explain the abrupt decrease in the ratio $\langle E_x \rangle_T / E_L$ as shown in Figure 3. The value of T_z^1 at which the knee of the curve occurs in Figure 3 corresponds to the value of T_{cz} in Figure 4 at which $\langle E_x \rangle_T$ for a given T begins to diverge from the unidirectional solution. At this point E_L continues to increase whereas $\langle E_x \rangle_T$ peaks and begins to decrease.

III-C.2 Characteristics of Dispersive Mass Transport in Oscillatory Flows

The physical reasoning behind the resonant behavior of $\langle E_x \rangle_T$ is straightforward and is begun by considering the extreme cases of $T \ll T_{cz}$ and $T \gg T_{cz}$:

(1) $T \ll T_{cz}$ - In this case the degree of vertical mixing is so small during one period of oscillation that an elemental volume initially residing in the water column at elevation z , and containing an initial concentration c , would remain at this elevation thus being transported over the closed pathline of flow, and returned to its initial position with no net longitudinal dispersion having occurred. Conversely, in a unidirectional flow with little or no vertical mixing the longitudinal dispersion would be very large.

(2) $T \gg T_{cz}$ - Here the rate of vertical mixing is so rapid that there is no time for the velocity shear profile to transport the substance longitudinally before it loses its identity through vertical mixing. In this case the oscillatory and unidirectional flow dispersive processes behave in a similar manner and are both small.

Thus, it is seen that for an oscillatory flow the longitudinal dispersive mass transport becomes small for both $T \ll T_{cz}$ and $T \gg T_{cz}$, whereas for a unidirectional flow the longitudinal dispersive mass transport varies directly with T_{cz} . This resonant nature of $\langle E_x \rangle_T$ can be

further explained on physical grounds by beginning with Case 1 above and gradually allowing the vertical mixing time to decrease. In this example the velocity shear profile in conjunction with the increased vertical mixing causes a net longitudinal mass transport to occur over a period of oscillation. This effect continues to increase as T_{cz} is decreased until the optimum ratio between the vertical mixing time and the period of oscillation is reached. At this point the longitudinal dispersive mass transport has reached its maximum value. A further decrease in T_{cz} begins to introduce the effect noted in Case 2 where the vertical mixing is now too rapid thus causing $\langle E_x \rangle_T$ to decrease.

Additional physical insight into the behavior of $\langle E_x \rangle_T$ can be obtained by considering the relative longitudinal dispersion associated with two oscillatory flows, one having twice the period of oscillation of the other. The two periods considered are 44,712 and 89,424 seconds corresponding to the periods of a semi-diurnal and diurnal tide respectively.

The first of the two comparisons is illustrated by Figures 5a and 5b. Once again the expression used for $\langle E_x \rangle_T$ is given by (III-56) with $u_{\max} = 1$ ft/sec for both the semi-diurnal and diurnal cases.

As shown in Figure 5a after an elapsed time of $t = 89,424$ sec the semi-diurnal tide will have completed two full cycles of excursion length L_1 , while the diurnal tide will have completed one full cycle of excursion length $L_2 = 2L_1$. In Figure 5b the favorable effect of the longer excursion length on longitudinal dispersion is seen for flows of different periods of oscillation with the same $u_{\max}$. The second comparison between the semi-diurnal and diurnal period flows uses excursion length as a normalizing parameter and is illustrated by Figures 6a and

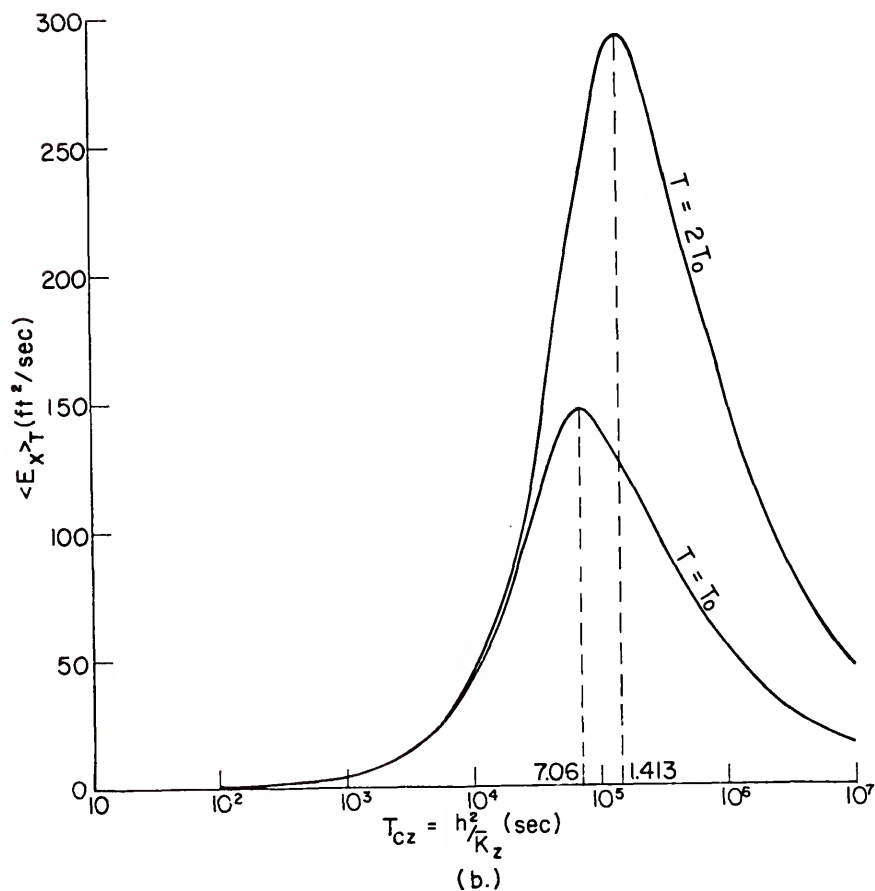
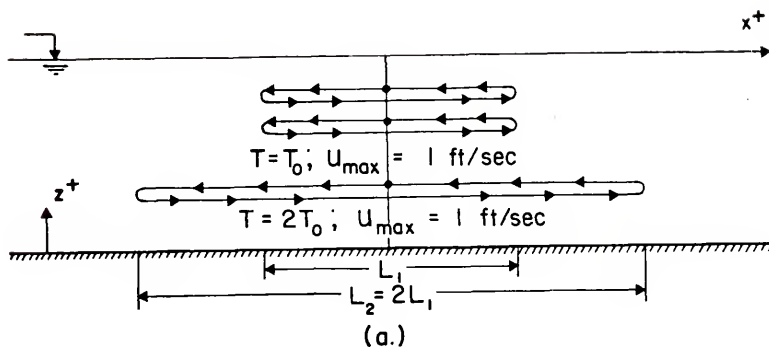


FIGURE 5 COMPARISON OF DISPERSION FOR DIURNAL AND SEMI-DIURNAL PERIODS OF OSCILLATION WITH SAME $u_{\max}$

6b. In this comparison the velocity for the diurnal tide is scaled downward so as to produce the same excursion length as the semi-diurnal tide. Figure 6a shows that after an elapsed time of $t = 89,424$ sec the semi-diurnal tide will have completed two full cycles of excursion length L_1 as before. However, the diurnal tide, because of the reduced velocity, has completed one full cycle of the same excursion length. Values of $\langle E_x \rangle_T$ are plotted vs. T_{cz} using (III-57) in Figure 6b. As seen from this plot the semi-diurnal dispersion has remained unchanged from Figure 5b whereas the diurnal curve is now everywhere less than the semi-diurnal curve. This is not surprising since from (III-57) the fixing of excursion length and vertical mixing time leaves only the decrease in the period of oscillation as a means of increasing longitudinal dispersion.

The results of this analysis on the predicted behavior of $\langle E_x \rangle_T$ in two dimensional oscillatory shear flow are summarized in non-dimensional form by Figure 7. In this figure the non-dimensional longitudinal dispersion coefficient as defined by (III-59) is plotted vs. T'_Z . Thus, by knowing the period of oscillation, the surface amplitude of velocity, the water depth, and the vertical eddy diffusivity, one could use Figure 7 to obtain a predicted value for the longitudinal dispersion coefficient. The resonant value of T'_Z for which $\langle E_x \rangle_T$ is a maximum is seen from Figure 7 to be

$$T'_Z = 1.58 \quad (\text{III-62})$$

This relationship is applicable to all oscillatory flows in infinitely wide rectangular channels provided the assumptions made in this analysis are reasonably well satisfied.

A look at the kinematic structure of the flow field and the

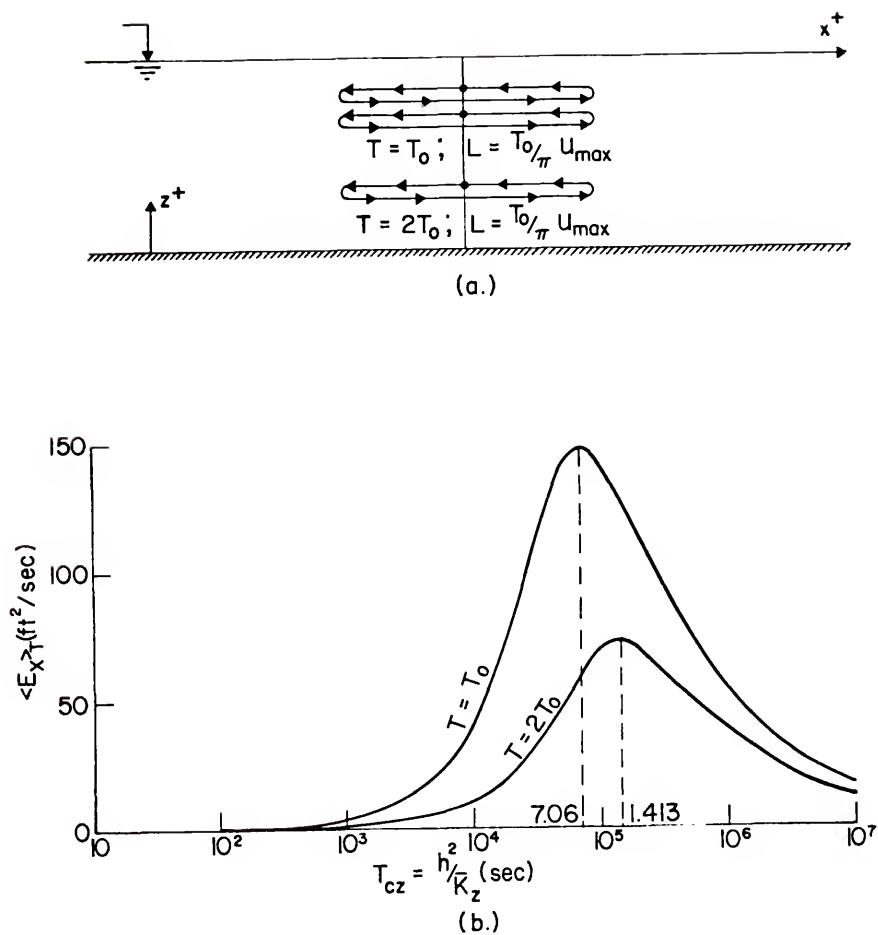


FIGURE 6 COMPARISON OF DISPERSION FOR DIURNAL AND SEMI-DIURNAL PERIODS OF OSCILLATION WITH SAME SURFACE EXCURSION LENGTH

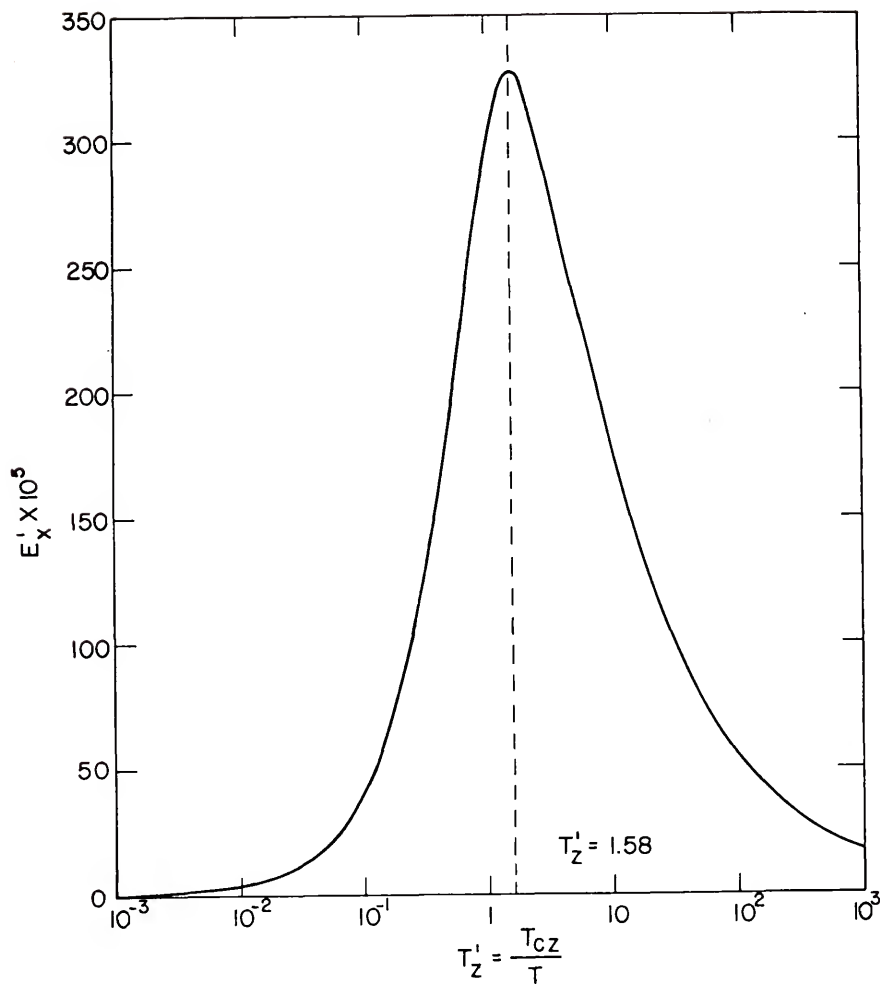


FIGURE 7 NON-DIMENSIONAL LONGITUDINAL DISPERSION COEFFICIENT E'_x , AS A FUNCTION OF T'_z FOR TWO DIMENSIONAL OSCILLATORY SHEAR FLOW

concentration distribution over depth is presented for three values of T'_Z in Figures 8, 9, and 10. In these figures, non-dimensional forms of the solutions for $u''(z,t)$ and $c''(z,t)$ given by (III-43) and (III-50) respectively are plotted over the non-dimensional depth, z/h . The non-dimensional forms used are

$$u''_{ND} = \frac{u''}{u_{\max}} \quad (\text{III-63})$$

$$c''_{ND} = \frac{c''}{\left(-\frac{\partial \bar{c}}{\partial \xi}\right)u_{\max}T} \quad (\text{III-64})$$

Each figure shows the u'' and c'' profiles for maximum flow and slack water. Several interesting points are illustrated here. First, the superior correlation of u'' and c'' over depth in Figure 9 when compared to the same correlation in Figures 8 and 10 demonstrates qualitatively why $\langle E_x \rangle_T$ is greater for $T'_Z = 1.58$ than for the other values of T'_Z considered. Second, the phase dependency of the velocity with depth is clearly shown in all figures near slack water with flow reversals occurring in the lower momentum layers near the channel bed prior to a shift in the upper higher momentum layers. Finally, it is interesting to note that the shear profile is confined to a decreasingly thinner layer near the bed with the upper portion of the profile becoming flatter as T'_Z is increased (increasing eddy viscosity by Reynolds analogy).

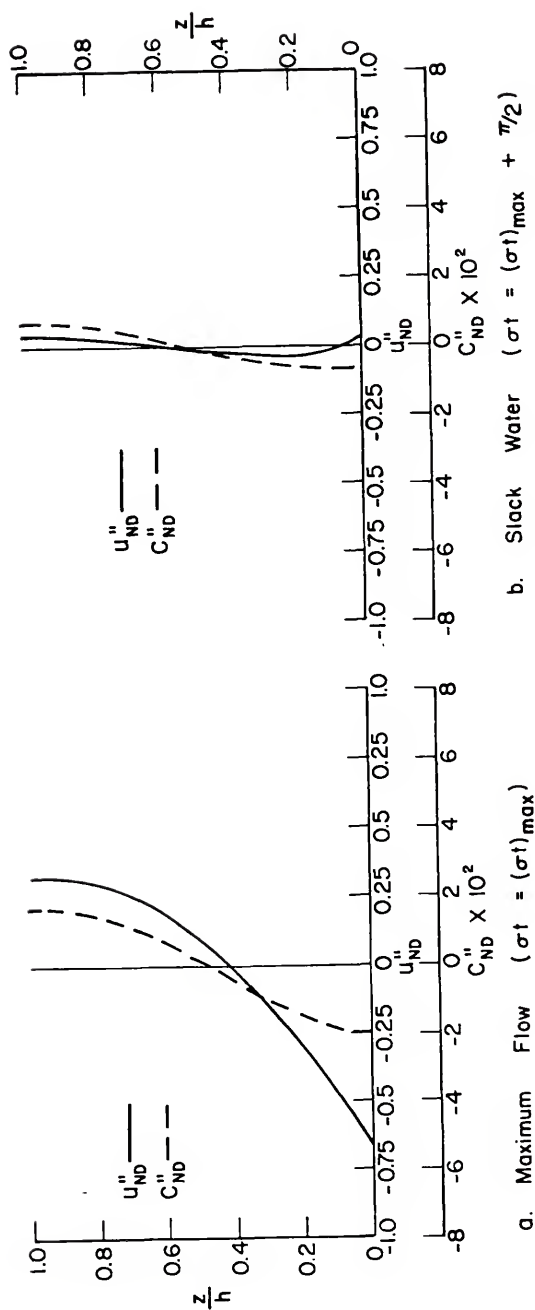


FIGURE 8 NON-DIMENSIONAL VELOCITY AND CONCENTRATION VARIATION
COMPONENT PROFILES FOR $T'_z = 0.5$

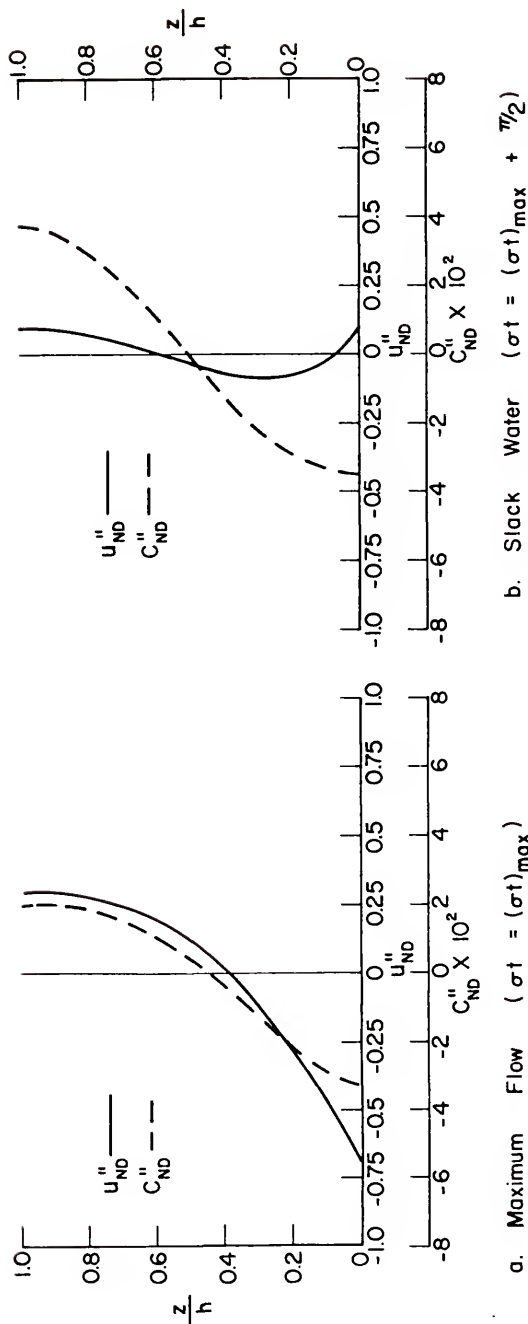


FIGURE 9 NON-DIMENSIONAL VELOCITY AND CONCENTRATION VARIATION
COMPONENT PROFILES FOR $T_z^1 = 1.58$

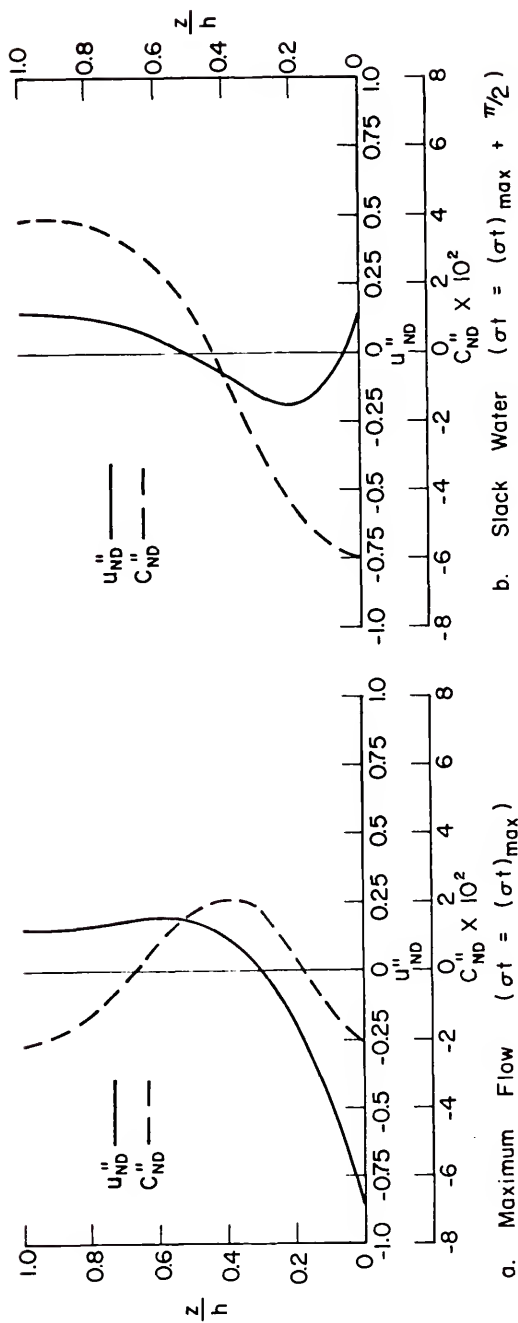


FIGURE 10 NON-DIMENSIONAL VELOCITY AND CONCENTRATION VARIATION
COMPONENT PROFILES FOR $T'_z = 5.0$

CHAPTER IV
DISPERSIVE MASS TRANSPORT IN RECTANGULAR
CHANNELS OF FINITE WIDTH

IV-A Oscillatory Flow

The physical problem and the general analytical approach used to solve it are the same in this chapter as those discussed in Chapter III with the added feature of a finite channel width. The effect of this feature is to introduce a three dimensional shear flow with vertical and transverse velocity variations which are used to force the transport diffusion equation to obtain a solution for $c''(y,z,t)$. Solutions for $u''(y,z,t)$ and $c''(y,z,t)$ are then correlated over the flow cross section and, in the case of oscillatory flow, averaged over the period of oscillation to obtain the longitudinal dispersive mass transport.

The coordinate system selected has its origin at the center of a rectangular section whose sides correspond to $z = \pm h$, and $y = \pm w/2$. As shown in Figure 11, the open rectangular channel is mathematically represented by the lower half of the section. Selection of the coordinate system in this manner preserves the symmetry of the problem about the origin and, as will be shown later, correctly predicts the same dispersive mass transport for equal degrees of skewness in channel geometry in either the vertical or lateral directions.

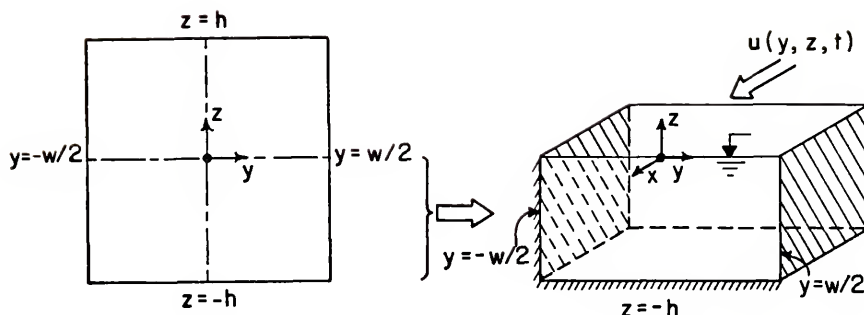


FIGURE 11. DEFINITION SKETCH FOR THREE DIMENSIONAL SHEAR FLOW

IV-A.1 Velocity Distribution

Using the same assumptions that were made in Chapter III regarding eddy viscosity, time averaging of turbulent fluctuations, and the form of the pressure gradient term, the equation of motion for uniform flow in the x direction may be written as

$$\frac{\partial u}{\partial t} = -K e^{i\sigma t} + \bar{\epsilon}_z \frac{\partial^2 u}{\partial z^2} + \bar{\epsilon}_y \frac{\partial^2 u}{\partial y^2} \quad (\text{IV-1})$$

If a solution of the form

$$u(y, z, t) = u_s(y, z) e^{i\sigma t} \quad (\text{IV-2})$$

is assumed to be valid for (IV-1) then the three dimensional shear flow boundary value problem for oscillatory flow may be formally stated as:

$$\text{PDE: } \bar{\epsilon}_z \frac{\partial^2 u_s}{\partial z^2} + \bar{\epsilon}_y \frac{\partial^2 u_s}{\partial y^2} - i\sigma u_s = K \quad (\text{IV-3})$$

$$\begin{aligned} \text{BC's: } u_s(y, \pm h) &= 0 \\ u_s(\pm \frac{w}{2}, z) &= 0 \end{aligned}$$

It is noted that physical considerations require that $u_s(y, z)$ be an even function in y and z. Thus, a solution to (IV-3) of the form

$$u_s(y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} \quad (\text{IV-4})$$

is assumed. An examination of (IV-4) with the statement of the boundary value problem above shows that the assumed solution satisfies the no slip conditions on the perimeter of the section. Moreover, (IV-4) also satisfies the condition of zero vertical shear at the free surface, and zero lateral shear at the centerline of the channel. The assumed solution is then substituted into (IV-3) and the Fourier coefficients, a_{mn} , are determined in the usual manner through the use of the orthogonal properties of the cosine function. Details of this analysis are presented in Appendix B. Solving for a_{mn} and thus determining the form of $u_s(y, z)$, the complete solution for the velocity function $u(y, z, t)$ using (IV-4) and (IV-2) is then

$$u(y, z, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{-16K(-1)^{m+n} \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} e^{i\sigma t}}{(2m+1)(2n+1)\pi^2 \{ \bar{\epsilon}_z \left[\frac{(2m+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2n+1)\pi}{w} \right]^2 + i\sigma \}} \quad (\text{IV-5})$$

To obtain the spatial variation, $u''(y, z, t)$, the cross-sectional mean velocity, $\bar{u}(t)$, must be subtracted out of (IV-5). The mean component of u is defined by

$$\bar{u}(t) = \frac{e^{i\sigma t}}{2hw} \int_{-h}^h \int_{-w/2}^{w/2} u_s(y, z) dy dz \quad (\text{IV-6})$$

Performing the integration in (IV-6) and subtracting the result from (IV-5) yields

$$u''(y, z, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \left\{ \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} - g(m, n) \right\} e^{i\sigma t} \quad (\text{IV-7})$$

where,

$$g(m,n) = \frac{4(-1)^{m+n}}{\pi^2(2m+1)(2n+1)} \quad (\text{IV-8})$$

and,

$$a_{mn} = \frac{-16K(-1)^{m+n}}{\pi^2(2m+1)(2n+1) \left\{ \bar{\epsilon}_z \left[\frac{(2m+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2n+1)\pi}{w} \right]^2 + i\sigma \right\}} \quad (\text{IV-9})$$

Since only the real portions of these expressions have any physical meaning, (IV-7) is transformed to polar form as was done in Chapter III for the two dimensional case. In addition, it will be assumed once again for simplification of results that Reynolds analogy applies thereby allowing the introduction of the vertical mixing time T_{cz} and the definition of a comparable lateral mixing time as

$$T_{cy} = \frac{(w/2)^2}{\bar{k}_y} \quad (\text{IV-10})$$

The solution for $u''(y,z,t)$ as a function of the pressure gradient may then be written as

$$u''(y,z,t) = -\frac{8KT}{\pi^3} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} e^{i(\sigma t - \alpha_1(m,n))}}{(2m+1)(2n+1) R_1^{\frac{1}{2}}(m,n)} \left\{ \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} - g(m,n) \right\} \quad (\text{IV-11})$$

where,

$$R_1(m,n) = \left\{ \frac{T}{2\pi T_{cz}} \left[\frac{(2m+1)\pi}{2} \right]^2 + \frac{T}{2\pi T_{cy}} \left[\frac{(2n+1)\pi}{2} \right]^2 \right\}^2 + 1 \quad (\text{IV-12})$$

$$\alpha_1(m,n) = \tan^{-1} \left\{ \frac{1}{\frac{T}{2\pi T_{cz}} \left[\frac{(2m+1)\pi}{2} \right]^2 + \frac{T}{2\pi T_{cy}} \left[\frac{(2n+1)\pi}{2} \right]^2} \right\} \quad (\text{IV-13})$$

In order to put (IV-11) in a more usable form, (IV-5) is used to obtain a relationship between the pressure gradient, K , and $u_{\max}$, where $u_{\max}$ is now defined as the amplitude of the periodic velocity function at the free surface centerline. Thus, from (IV-5)

$$u(0,0,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{-16 K(-1)^{m+n} e^{i\sigma t}}{\pi^2 (2m+1)(2n+1) \left\{ \bar{\epsilon}_z \left[\frac{(2m+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2n+1)\pi}{w} \right]^2 + i\sigma \right\}} \quad (\text{IV-14})$$

which upon maximization with respect to time yields the relationship

$$K = - \frac{u_{\max} \pi^3}{8T\phi} \quad (\text{IV-15})$$

where,

$$\phi = \left[\left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} (R_1(m,n)-1)^{\frac{1}{2}}}{(2m+1)(2n+1) R_1(m,n)} \right\}^2 + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1) R_1(m,n)} \right\}^2 \right]^{\frac{1}{2}} \quad (\text{IV-16})$$

The details of developing (IV-15) from (IV-14) are presented in Appendix C. Substitution of (IV-15) into (IV-11) for K yields the following expression for u'' as a function of $u_{\max}$

$$u''(y,z,t) = \frac{u_{\max}}{\phi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} e^{i(\sigma t - \alpha_1(m,n))}}{(2m+1)(2n+1) R_1^{\frac{1}{2}}(m,n)} \left\{ \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} - g(m,n) \right\} \quad (\text{IV-17})$$

IV-A.2 Concentration Distribution

The form of the transport diffusion equation used to solve for $c''(y,z,t)$ is identical to (II-27), or

$$\frac{\partial c''}{\partial t} - \bar{K}_z \frac{\partial^2 c''}{\partial z^2} - \bar{K}_y \frac{\partial^2 c''}{\partial y^2} = -u'' \frac{\partial \bar{c}}{\partial \xi} \quad (\text{IV-18})$$

where $u''(y,z,t)$ is given by (IV-7). It is assumed that $c''(y,z,t)$ is of the form

$$c''(y,z,t) = c''_s(y,z) e^{i\sigma t} \quad (\text{IV-19})$$

where $c''_s(y,z)$ is complex following the same approach used with $u''(y,z,t)$. Substituting (IV-19) into (IV-18) and requiring zero diffusive flux across the boundaries of the flow, the boundary value problem is then stated as

$$\text{PDE: } \bar{K}_z \frac{\partial^2 c''_s}{\partial z^2} + \bar{K}_y \frac{\partial^2 c''_s}{\partial y^2} - i\sigma c''_s = u''_s \frac{\partial \bar{c}}{\partial \xi} \quad (\text{IV-20})$$

$$\text{BC's: } \left. \frac{\partial c''_s}{\partial z} \right|_{z = \pm h} = 0$$

$$\left. \frac{\partial c''_s}{\partial y} \right|_{y = \pm w/2} = 0$$

Now, c''_s must also be an even function in y and z based on physical arguments and is therefore assumed to be of the form

$$c''_s(y,z) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} a'_{\ell k} \cos \frac{\ell \pi z}{h} \cos \frac{2k \pi y}{w} \quad (\text{IV-21})$$

Equation (IV-21) is seen to satisfy the conditions of zero diffusive flux across the flow boundaries. Substituting for c''_s using (IV-21) and for u''_s using (IV-4) and (IV-7), (IV-20) becomes

$$\sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \{ \bar{K}_z \left(\frac{\ell\pi}{h} \right)^2 + \bar{K}_y \left(\frac{2k\pi}{w} \right)^2 + i\sigma \} a'_{\ell k} \cos \frac{\ell\pi z}{h} \cos \frac{2k\pi y}{w} =$$

$$- \left(\frac{\partial \bar{C}}{\partial \xi} \right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \left\{ \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} - g(m,n) \right\} \quad (\text{IV-22})$$

where a_{mn} is given by (IV-9). The unknown Fourier coefficients $a'_{\ell k}$ are solved for in the same manner as was used to determine a_{mn} . The detailed mathematics of the solution are presented in Appendix B, the result of which is given by

$$a'_{\ell k} = \left(\frac{\partial \bar{C}}{\partial \xi} \right) \frac{256K}{\pi^4 \{ \bar{K}_z \left(\frac{\ell\pi}{h} \right)^2 + \bar{K}_y \left(\frac{2k\pi}{w} \right)^2 + i\sigma \}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \quad (\text{IV-23})$$

$$\frac{(-1)^{-(p+q)}}{[(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2] \{ \bar{\epsilon}_z \left[\frac{(2p+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2q+1)\pi}{w} \right]^2 + i\sigma \}}$$

Thus, the solution for $c''(y,z,t)$ includes four infinite summations arising from the non-orthogonal nature of the cosine functions representing $u''_s(y,z)$ and $c''_s(y,z)$. Again introducing Reynolds analogy and the expression for the vertical and transverse mixing times, the solution for $c''(y,z,t)$ in polar form as a function of the pressure gradient is

$$c''(y,z,t) = \left(\frac{\partial \bar{C}}{\partial \xi} \right) \frac{64KT^2}{\pi^6} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty}$$

$$\frac{\mu_{\ell k} (-1)^{-(\ell+k)} \cos \frac{\ell\pi z}{h} \cos \frac{2k\pi y}{w} e^{i\theta_2}}{[(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2] R_1^{\frac{1}{2}}(p,q) R_2^{\frac{1}{2}}(\ell,k)} \quad (\text{IV-24})$$

where,

$$\theta_2 = \sigma t - \alpha_1(p, q) - \alpha_2(\ell, k) \quad (\text{IV-25})$$

$R_1(p, q)$ is given by (IV-12) with p replacing m and q replacing n .

$$R_2(\ell, k) = \left\{ \frac{T}{2\pi T_{cz}} (\ell\pi)^2 + \frac{T}{2\pi T_{cy}} (k\pi)^2 \right\}^2 + 1 \quad (\text{IV-26})$$

$\alpha_1(p, q)$ is given in (IV-13) with p replacing m and q replacing n .

$$\alpha_2(\ell, k) = \tan^{-1} \left\{ \frac{1}{\frac{T}{2\pi T_{cz}} (\ell\pi)^2 + \frac{T}{2\pi T_{cy}} (k\pi)^2} \right\} \quad (\text{IV-27})$$

$$\mu_{\ell k} = \begin{cases} 0, & k = \ell = 0 \\ \frac{1}{2}, & k = 0 \text{ or } \ell = 0 \\ 1, & k \neq 0, \ell \neq 0 \end{cases} \quad (\text{IV-28})$$

The factor $\mu_{\ell k}$ is necessarily included in (IV-24) to properly account for the a'_{0k} and $a'_{\ell 0}$ terms, and to insure that $c''(y, z, t)$ has a zero mean over the flow cross-sectional area. Detailed mathematics for arriving at (IV-24) are given in Appendix B.

The expression of $c''(y, z, t)$ as a function of $u_{\max}$ is determined by substitution for K in (IV-24) using (IV-15) to yield,

$$c''(y, z, t) = \left(-\frac{\partial \bar{c}}{\partial \xi} \right) \frac{8u_{\max} T}{\pi^3 \phi} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k} (-1)^{-(\ell+k)} \cos \frac{\ell\pi z}{h} \cos \frac{2k\pi y}{w} e^{i\theta_2}}{[(2p+1)^2 - (2\ell)^2] [(2q+1)^2 - (2k)^2] R_1^{\frac{1}{2}}(p, q) R_2^{\frac{1}{2}}(\ell, k)} \quad (\text{IV-29})$$

IV-A.3 Dispersive Mass Transport

The time mean dispersive mass transport averaged over one period of oscillation for the full rectangular section as shown in Figure 11 is given by

$$\langle \dot{m} \rangle_T = \left\langle \int_{-w/2}^{w/2} \int_{-h}^h R_e[u''(y,z,t)] \cdot R_e[c''(y,z,t)] dz dy \right\rangle_T \quad (IV-30)$$

The integrations in (IV-30) are carried out in Appendix B and it is noted that the non-orthogonality of the cosine functions representing u'' and c'' , arising from the form of their respective arguments, once again produces two additional infinite summations. Thus, using (IV-11) and (IV-24) for u'' and c'' the following expression for the longitudinal dispersive mass transport over the region $-h \leq z \leq h$ and $-w/2 \leq y \leq w/2$ is determined

$$\langle \dot{m} \rangle_T = \left(- \frac{\partial \bar{c}}{\partial \xi} \right) \frac{2048 \, wh \, K^2 T^3}{\pi^{11}} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k} \{ (R_2(\ell, k) - 1)^{\frac{1}{2}} \cos \lambda + \sin \lambda \}}{\delta_1(\ell, k, m, n, p, q)} \quad (IV-31)$$

where,

$$\lambda = \alpha_1(m, n) - \alpha_1(p, q) \quad (IV-32)$$

$$\delta_1(\ell, k, m, n, p, q) = [(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2][(2n+1)^2 - (2k)^2] \cdot$$

$$[(2m+1)^2 - (2\ell)^2] R_1^{\frac{1}{2}}(m, n) R_1^{\frac{1}{2}}(p, q) R_2(\ell, k) \quad (IV-33)$$

Now the dispersive mass transport associated with the open rectangular channel comprising the lower half of the full box is $\frac{1}{2}\dot{m}_T$. Therefore, the Fickian flux representation of the dispersive mass transport in the open rectangular channel is simply

$$- \langle E_x \rangle_T (wh) \frac{\partial \bar{C}}{\partial \xi} = \frac{1}{2} \dot{m}_T \quad (\text{IV-34})$$

Replacing $\dot{m}_T$ in (IV-34) with the right hand side of (IV-31) and solving for $\langle E_x \rangle_T$ yields

$$\langle E_x \rangle_T = \frac{1024K^2T^3}{\pi^{11}} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k} \{ (R_2(\ell, k) - 1)^{\frac{1}{2}} \cos \lambda + \sin \lambda \}}{\delta_1(\ell, k, m, n, p, q)} \quad (\text{IV-35})$$

The longitudinal dispersion coefficient is expressed as a function of $u_{\max}$ by substituting for K in (IV-35) using (IV-15) to give

$$\langle E_x \rangle_T = \frac{16Tu_{\max}^2}{\pi^5\phi^2} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k} \{ (R_2(\ell, k) - 1)^{\frac{1}{2}} \cos \lambda + \sin \lambda \}}{\delta_1(\ell, k, m, n, p, q)} \quad (\text{IV-36})$$

To facilitate the presentation of results and the application of (IV-36) for predictive purposes a non-dimensional relative mixing time, T'_C , is introduced as

$$T'_C = \frac{T'_Z}{T'_Y} = \frac{T_{CZ}}{T_{CY}} \quad (\text{IV-37})$$

where,

$$T'_y = \frac{T_{cy}}{T} \quad (\text{IV-38})$$

Thus, as shown by (IV-37), T'_c is a measure of the relative effects of vertical and lateral shear for a given width-to-depth ratio, and vertical and lateral eddy diffusivities. With the aid of (IV-37) the following definitions can then be made:

$$r_1(m,n) = (2\pi T'_z)^2 R_1(m,n) = \left\{ \left[\frac{(2m+1)\pi}{2} \right]^2 + T'_c \left[\frac{(2n+1)\pi}{2} \right]^2 \right\}^2 + (2\pi T'_z)^2 \quad (\text{IV-39})$$

$$r_2(\ell,k) = (2\pi T'_z)^2 R_2(\ell,k) = \{ (\ell\pi)^2 + T'_c (k\pi)^2 \}^2 + (2\pi T'_z)^2 \quad (\text{IV-40})$$

$$\alpha_1(m,n) = \tan^{-1} \left\{ \frac{2\pi T'_z}{\left[\frac{(2m+1)\pi}{2} \right]^2 + T'_c \left[\frac{(2n+1)\pi}{2} \right]^2} \right\} \quad (\text{IV-41})$$

$$\begin{aligned} \psi^2 = \frac{\phi^2}{(2\pi T'_z)^2} = & \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} [r_1(m,n) - (2\pi T'_z)^2]^{\frac{1}{2}}}{(2m+1)(2n+1)r_1(m,n)} \right\}^2 \\ & + \left\{ 2\pi T'_z \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1)r_1(m,n)} \right\}^2 \end{aligned} \quad (\text{IV-42})$$

These relationships are then applied to (IV-36) to obtain an expression for the non-dimensional longitudinal dispersion coefficient, E'_x , defined

by (III-58). This expression represents the final form of the solution for the case of three dimensional oscillatory shear flow and is therefore presented along with a summary of the associated terms as follows:

$$E'_X = \frac{\langle E_X \rangle_T}{u_{\max}^2 T} = \frac{32T'_Z}{\pi^4 \psi^2} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k} \{ [r_2(\ell, k) - (2\pi T'_Z)^2]^{\frac{1}{2}} \cos \lambda + (2\pi T'_Z) \sin \lambda \}}{\delta_2(\ell, k, m, n, p, q)} \quad (\text{IV-43})$$

where,

$$\begin{aligned} \delta_2(\ell, k, m, n, p, q) &= (2\pi T'_Z)^4 \delta_1(\ell, k, m, n, p, q) \\ &= [(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2][(2m+1)^2 - (2\ell)^2] \cdot \\ &\quad [(2n+1)^2 - (2k)^2] r_1^{\frac{1}{2}}(m, n) r_1^{\frac{1}{2}}(p, q) r_2(\ell, k) \end{aligned} \quad (\text{IV-44})$$

$$r_1(m, n) = \left\{ \left[\frac{(2m+1)\pi}{2} \right]^2 + T_C' \left[\frac{(2n+1)\pi}{2} \right]^2 \right\}^2 + (2\pi T'_Z)^2$$

$$r_1(p, q) = r_1(m, n) \text{ with } p \text{ replacing } m \text{ and } q \text{ replacing } n$$

$$r_2(\ell, k) = \left\{ (\ell\pi)^2 + T_C' (k\pi)^2 \right\}^2 + (2\pi T'_Z)^2$$

$$\alpha_1(m, n) = \tan^{-1} \left\{ \frac{2\pi T'_Z}{\left[\frac{(2m+1)\pi}{2} \right]^2 + T_C' \left[\frac{(2n+1)\pi}{2} \right]^2} \right\}$$

$$\alpha_1(p, q) = \alpha_1(m, n) \text{ with } p \text{ replacing } m \text{ and } q \text{ replacing } n$$

$$\begin{aligned} \psi^2 &= \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} [r_1(m, n) - (2\pi T'_Z)^2]^{\frac{1}{2}}}{(2m+1)(2n+1)r_1(m, n)} \right\}^2 \\ &\quad + \left\{ 2\pi T'_Z \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1)r_1(m, n)} \right\}^2 \end{aligned}$$

$$\lambda = \alpha_1(m,n) - \alpha_1(p,q)$$

$$\mu_{\ell k} = \begin{cases} 1 & , \quad k \neq 0, \quad \ell \neq 0 \\ \frac{1}{2} & , \quad k = 0 \text{ or } \ell = 0 \\ 0 & , \quad k = \ell = 0 \end{cases}$$

The symmetry of this solution can now be seen. If, for example, T'_C had been inversely defined as T_{cy}/T_{cz} , then (IV-36) would have reduced to (IV-43) with T'_y replacing T'_z everywhere, and the newly defined T'_C multiplying all complementary terms in r_1 , r_2 , and α_1 . This form of (IV-43) would predict the same value for E'_x as the original formulation provided that $u_{\max}$ and T were the same and that the new T'_y equaled the old T'_z and the new T'_C equaled the old T'_C . Stated another way, for the same $u_{\max}$ and T , a channel whose half-width was twice its depth would produce the same dispersive mass transport as a channel whose depth was twice its half-width provided that the ratio $\bar{K}_y/\bar{K}_z$ for the first case equaled $\bar{K}_z/\bar{K}_y$ in the second case.

The functional behavior of E'_x as given by (IV-43) for three dimensional oscillatory shear flow is illustrated by Figure 12. In this figure E'_x is plotted against the non-dimensional vertical mixing time, T'_z , for fixed values of the relative mixing time, T'_C . Because the solution for E'_x contains six infinite summations nested in series some limitations were necessary in carrying out the required computations. By varying the upper bound on each of the sums it was demonstrated that the solution converges upward to its limiting value. The convergence occurs reasonably rapidly; however, above an upper limit of 10 terms for each sum the rate of convergence is slowed considerably. The results

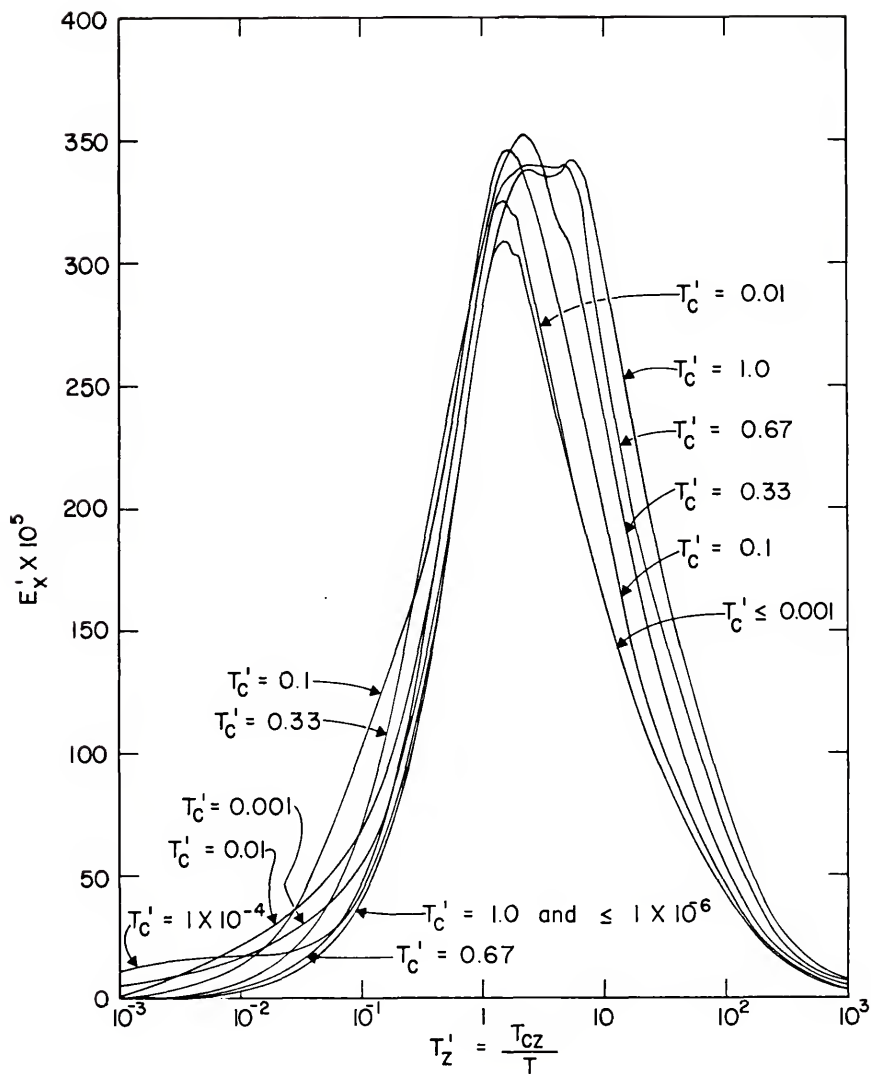


FIGURE 12 NON-DIMENSIONAL LONGITUDINAL DISPERSION COEFFICIENT E'_x , AS A FUNCTION OF T'_z FOR FULL RANGE OF T'_c VALUES

presented in Figure 12 and in subsequent figures were computed using an upper bound of 10 on each sum; therefore, each value of E'_x includes approximately 1.2×10^6 terms. An estimate of the error between the computed values of E'_x and the values predicted by the complete series represented by (IV-43) is obtained by comparing the limiting curve for a very wide channel in Figure 12 with the corresponding curve for an infinitely wide channel shown in Figure 7. The limiting curve in Figure 12 is identified by $T'_c \leq 1 \times 10^{-6}$ on the left hand side of the peak and by $T'_c \leq 0.001$ on the right hand side of the peak. A comparison of the two curves suggests that for $T'_c \leq 1 \times 10^{-6}$ the solution for E'_x given by (IV-43) very nearly approximates for all T'_z the solution given by (III-59) for E'_x in an infinitely wide channel. The general shapes of the two curves are nearly identical with both peaks occurring at $T'_z = 1.58$. For the infinitely wide channel the peak value of E'_x is 3.27×10^{-3} whereas for the limiting curve in Figure 12 it is 3.07×10^{-3} or 6 per cent below the infinitely wide case. Thus, it appears that the three dimensional solution for E'_x approaches the two dimensional solution for E'_x in the limit as $T'_c \rightarrow 0$, and it is estimated that values predicted for E'_x in Figure 12 are low but generally within 10 per cent of their actual values as determined by (IV-43). This does not apply to the range $T'_z > 100$ where the Figure 12 values are as much as 70 per cent low. However, this is not a region of practical significance since values of T'_z in nature are generally < 100 .

To aid in the interpretation of results, the eight curves shown in Figure 12 have been separated into two groups and are replotted in Figures 13 and 14. Figure 13 shows the E'_x resonance curves for $T'_c = 1.0, 0.67, 0.33$, and 0.1 while Figure 14 shows the corresponding curves for

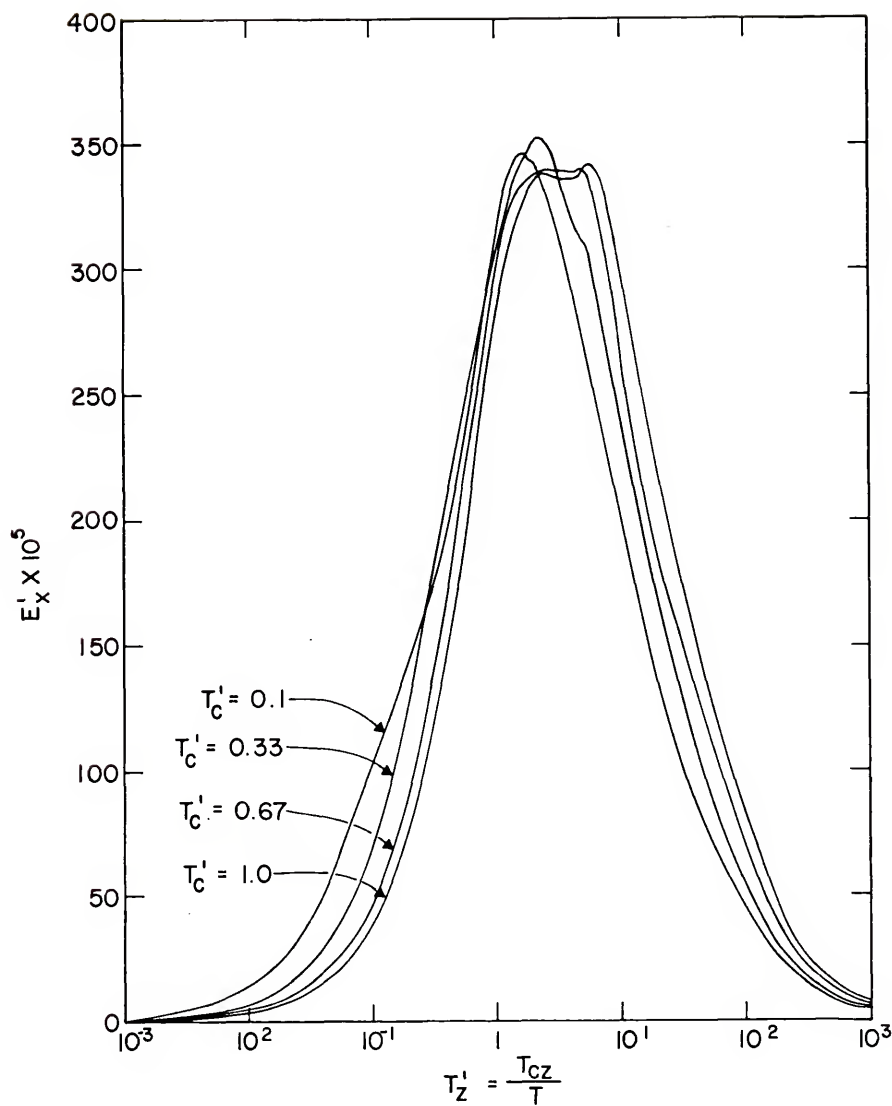


FIGURE 13 NON-DIMENSIONAL LONGITUDINAL DISPERSION COEFFICIENT E'_x AS A FUNCTION OF T'_z FOR $T'_c = 1.0, 0.67, 0.33$ AND 0.1

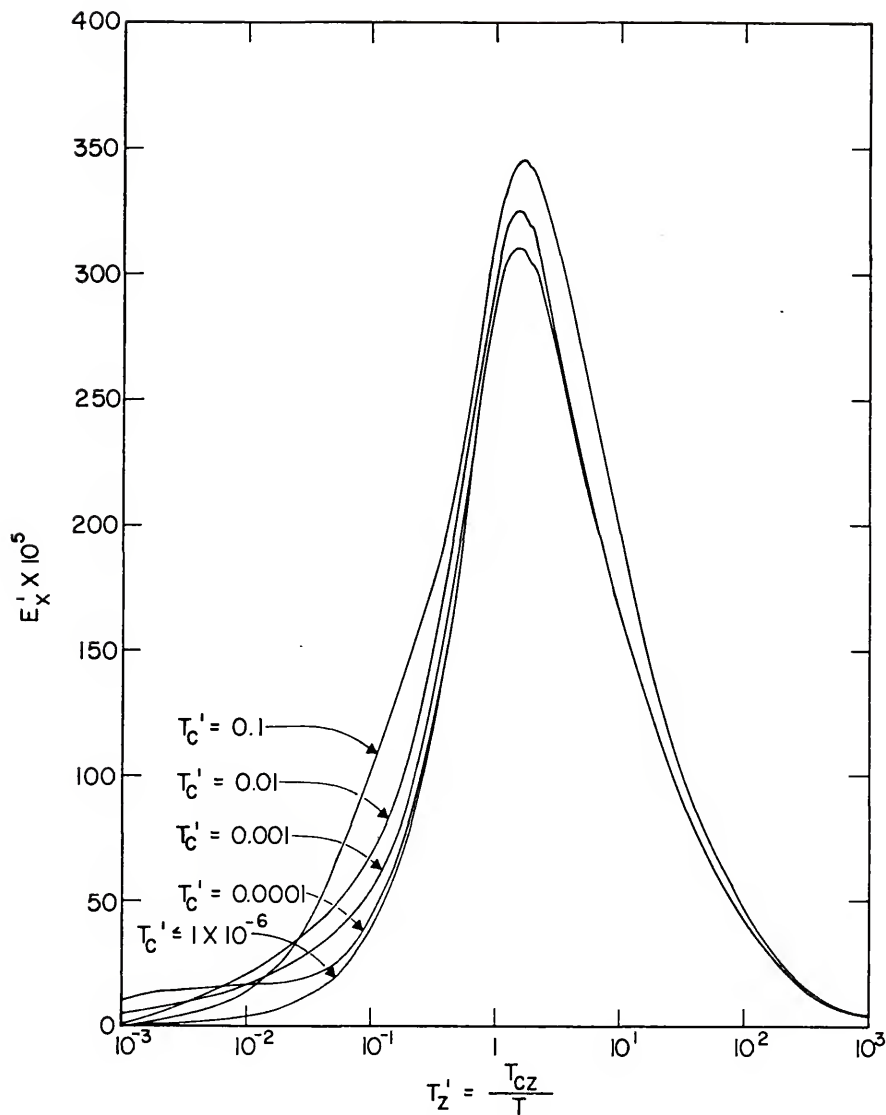


FIGURE 14 NON-DIMENSIONAL LONGITUDINAL DISPERSION COEFFICIENT E'_x AS A FUNCTION OF T'_z FOR $T'_c = 0.1, 0.01, 0.001, 0.0001$, and $\leq 1 \times 10^{-6}$

$T'_C = 0.1, 0.01, 0.001, 0.0001$, and $\leq 1 \times 10^{-6}$. Note that the curve for $T'_C = 0.1$ is repeated to provide a common reference. If for the purposes of interpretation it is assumed that $\bar{K}_Z = \bar{K}_Y$ then T'_C may be written as

$$T'_C = \left[\frac{h}{(w/2)} \right]^2 \quad (\text{IV-45})$$

which is simply the inverse square of the channel half-width to depth ratio. Referring to Figures 12, 13, and 14 with the aid of (IV-45) it is seen that as the half-width of the channel is increased from a value equal to the depth the resonant peak shifts from $T'_Z = 4.5$ leftward toward $T'_Z = 1.58$ at which it occurs when $T'_C \leq 0.001$. Concurrent with the shift is an initial increase in the peak value of E'_X as T'_C is decreased from 1.0 to 0.25 followed by a decrease in the peak value as T'_C is further decreased to 0.001. Decreasing T'_C below this value has no effect on E'_X for $T'_Z > 1.2$. A summary of the maximum value of E'_X achieved and the value of T'_Z at which it occurs for the corresponding T'_C is given in Table 1. These data are also presented graphically in Figures 15 and 16.

TABLE 1 PREDICTED RESONANT PEAK DATA FOR
LONGITUDINAL DISPERSION COEFFICIENT
IN THREE DIMENSIONAL OSCILLATORY SHEAR FLOW

Relative Mixing Time, T'_C	Non-Dimensional Vertical Mixing Time, T'_Z at which Maximum Occurs	Peak Value of Non-Dimensional Longitudinal Dispersion Coefficient, $E'_X \times 10^5$
1.00	4.50	339
0.67	3.75	340
0.40	2.40	350
0.33	2.20	353
0.25	1.95	354
0.20	1.80	352.5
0.10	1.65	346
0.01	1.60	325
0.001	1.58	310
0.0001	1.58	307
0.00001	1.58	307
0.000001	1.58	307

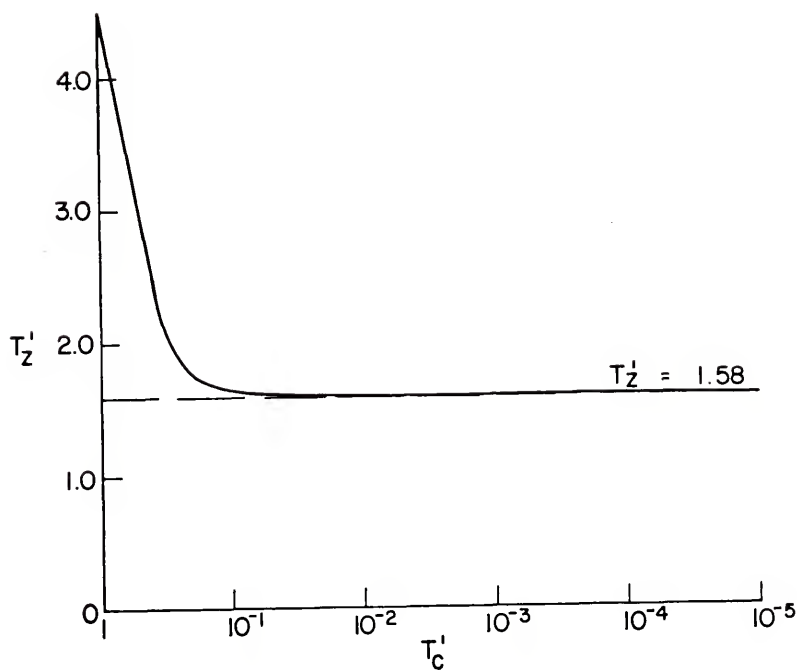


FIGURE 15 NON-DIMENSIONAL VERTICAL MIXING TIME, T'_z , AT WHICH MAXIMUM E'_x OCCURS AS A FUNCTION OF T'_c

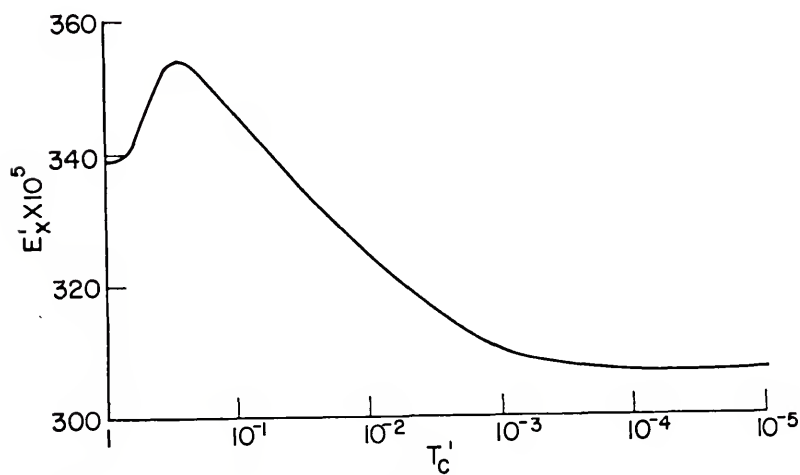


FIGURE 16 PEAK E'_x AS A FUNCTION OF T'_c

The effect of T'_C on the general shape of the resonance curve for E'_X is complex. To understand this behavior it is best to treat the right and left hand sides of each curve separately and to note that for the two extreme channel geometries corresponding to $T'_C = 1.0$ and $T'_C \leq 1 \times 10^{-6}$ the left hand sides of the respective curves are coincident up to $T'_Z = 1.0$. Looking at the right hand side of the curves in Figure 12 it is seen that as T'_C is decreased from 1.0 to 0.001 the curves are depressed toward the limiting two dimensional curve, merging with it at correspondingly lower values of T'_Z as T'_C is decreased.

For values of $T'_C \leq 0.001$ the right hand side of the curve remains unchanged. Thus, for values of $T'_Z > 7$ the effect of lateral shear on longitudinal dispersive mass transport in oscillatory flow is greatest for width-to-depth ratios corresponding to $T'_C = 1.0$ and decreases as the width-to-depth ratio is increased to a value corresponding to $T'_C = 0.001$. For $T'_C < 0.001$ the lateral shear has no effect and the dispersive mechanism behaves as a two dimensional shear flow phenomenon.

The behavior of the left hand side of the E'_X resonance curves is most easily seen from Figures 13 and 14. As T'_C is initially decreased from 1.0 to 0.33 the curve shifts to the left thereby predicting higher values of E'_X for corresponding values of T'_Z . Decreasing T'_C beyond this point, however, causes the upper portion of the curve to shift back toward the right predicting lower values of E'_X while the lower portion of the curve is flattened predicting higher values of E'_X for small values of T'_Z . Referring to Figure 14, as T'_C is decreased from 0.1 to 1×10^{-6} the upper portion of the curve continues to be suppressed toward the right and merges with the limiting two dimensional curve at correspondingly lower values of T'_Z while the lower extreme portion of

the curve continues to flatten until it also merges with the limiting two dimensional curve. For $T'_C \leq 1 \times 10^{-6}$ the left hand side remains unchanged.

It is observed that provided one knows the values of the non-dimensional vertical and lateral mixing times, T'_Z and T'_y , it is possible to predict the limiting width-to-depth ratio for which the three dimensional oscillatory flow case reduces to the more easily applied two dimensional case. The general condition for which this is true may be stated as

$$T'_C \leq T'_{C\infty} \quad (IV-46)$$

where $T'_{C\infty}$ is the limiting value of T'_C for which the two dimensional oscillatory dispersion solution is valid. Using the definitions of T'_C , T'_{CZ} , and T'_{cy} (IV-46) can then be written as

$$T'_{C\infty} \geq \left(\frac{4h^2}{w^2} \right) \left(\frac{\bar{K}_y}{\bar{K}_z} \right)$$

or,

$$\frac{w}{h} \geq 2 \sqrt{\frac{\bar{K}_y}{T'_{C\infty} \bar{K}_z}} \quad (IV-47)$$

It is noted from (IV-47) that increases in $\bar{K}_y$ due to secondary currents and horizontal eddies produced by channel irregularities will increase the value of w/h for which the two dimensional solution applies. Similarly, decreases in $\bar{K}_z$ due to density stratification will have the same effect. Values of $T'_{C\infty}$ are obtained from Figure 12. It is assumed that the channel is straight and that the fluid is homogeneous with no density stratification. For these conditions Ward (23) has shown in a detailed set of laboratory experiments that the time averaged ratio

of $\frac{\bar{k}_y}{\bar{k}_z}$ in an oscillatory flow is approximately twice the value determined by Elder (4) for unidirectional flow. Using the mean of four experimental runs from Ward's data it is assumed that

$$\frac{\bar{k}_y}{\bar{k}_z} = 6.6 \quad (\text{IV-48})$$

The predicted limiting width-to-depth ratios based upon these assumptions are presented in Table 2.

TABLE 2 LIMITING WIDTH-TO-DEPTH RATIOS FOR APPLICATION OF THE OSCILLATORY TWO DIMENSIONAL SHEAR FLOW DISPERSION COEFFICIENT

T'_z values for which valid	T'_c values for which valid	Width-to-depth Ratio, (w/h)
≥ 6.0	≤ 0.01	≥ 51
≥ 1.2	≤ 0.001	≥ 162
≥ 0.35	≤ 0.0001	≥ 514
≥ 0.04	≤ 0.00001	≥ 1625
$\pm \infty$	≤ 0.000001	≥ 5138
≤ 1.2	1.0	5.1

For given values of T'_z and T'_c Table 2 is used to determine if (w/h) is large enough to allow the use of the more simple solution for E'_x presented in III-B.

Before passing from the present section it is re-emphasized that while the preceding remarks have been directed toward channel geometries in which the half-width is always equal to or greater than the depth, they apply equally to channel geometries in which the depth is always

equal to or greater than the half-width provided the conditions for symmetry previously discussed are met. Thus, Figures 12, 13, and 14 could be used equally well for cases where $\frac{w}{2h} < 1$ provided T'_z were replaced with T'_y and T'_c were redefined as T'_y/T'_z . Perhaps the best way to show the symmetry of the solution about $T'_c = 1$ is by means of a plot such as Figure 17. In this figure isolines of E'_x for channel geometries skewed both in depth and width are plotted. The lower half of this figure corresponds to geometries in which $\frac{w}{2h} \geq 1$ whereas the upper half corresponds to geometries in which $\frac{w}{2h} \leq 1$, provided $\bar{k}_z = \bar{k}_y$.

- ⊙ Maximum Value of $E'_x = 354 \times 10^{-5}$ occurring at $T'_c = 0.25$ and $T'_z = 1.95$

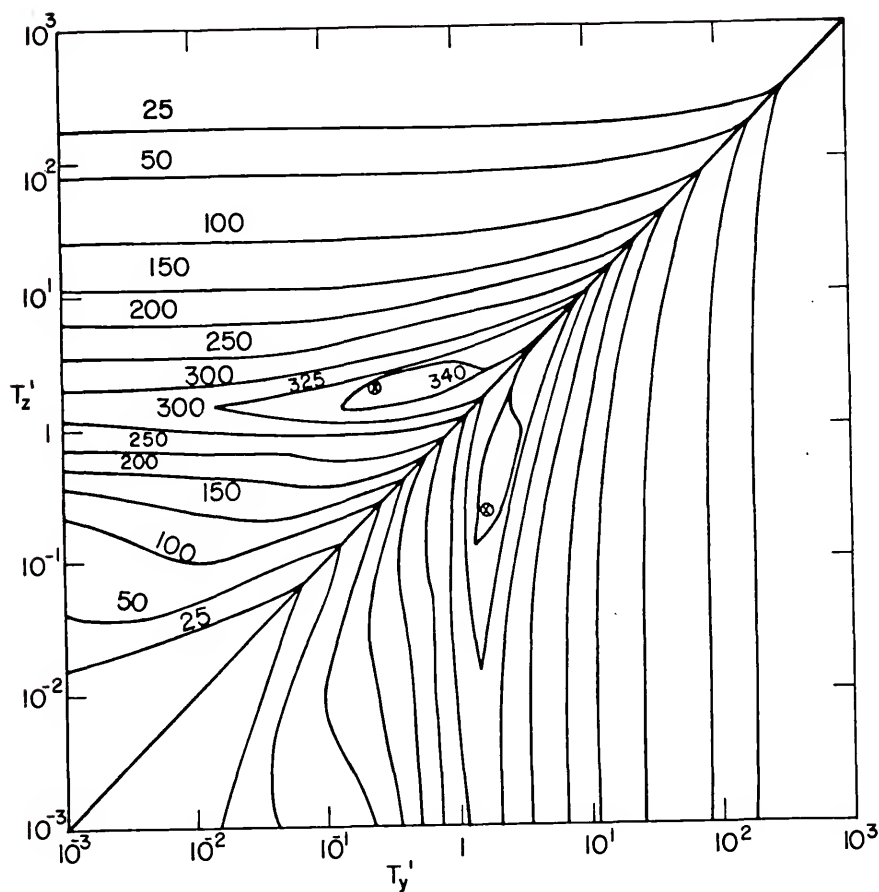


FIGURE 17 SYMMETRIC DISPLAY OF ISOLINES OF $E'_x \times 10^5$
IN THREE DIMENSIONAL OSCILLATORY SHEAR FLOW

IV-B Steady Unidirectional Flow

The following analysis is simplified by using, where applicable, the results of the previous section.

IV-B.1 Velocity Distribution

The coordinate system used and the boundary value problem formulation for the steady flow case are identical to that described in IV-A except for the exclusion of the local acceleration term. Thus, for no slip conditions on the perimeter of the full rectangular section and a constant pressure gradient as defined by (III-2) the flow boundary value problem for $u(y,z)$ is

$$\text{PDE:} \quad \bar{\epsilon}_z \frac{\partial^2 u}{\partial z^2} + \bar{\epsilon}_y \frac{\partial^2 u}{\partial y^2} = K \quad (\text{IV-49})$$

$$\text{BC's:} \quad u(\pm w/2, z) = 0$$

$$u(y, \pm h) = 0$$

Proceeding in the same manner as before, assume a solution of the form

$$u(y,z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} \quad (\text{IV-50})$$

which satisfies the boundary conditions and provides for zero vertical shear stress at the free surface and zero lateral shear stress at the centerline of the channel. Equation (IV-50) is then substituted into (IV-49) to yield

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{ \bar{\epsilon}_z \left[\frac{(2m+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2n+1)\pi}{w} \right]^2 \} \cdot$$

$$a_{mn} \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} = -K$$

(IV-51)

A comparison of (IV-51) with (B-3) in Appendix B shows them to be of the same general form. Therefore, the solutions for $u(y,z)$ and $u''(y,z)$ may be written directly as

$$u(y,z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{-16K(-1)^{m+n} \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w}}{\pi^2 (2m+1)(2n+1) \{ \bar{\epsilon}_z \left[\frac{(2m+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2n+1)\pi}{w} \right]^2 \}} \quad (\text{IV-52})$$

and,

$$u''(y,z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \left\{ \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} - g(m,n) \right\} \quad (\text{IV-53})$$

where,

$$a_{mn} = \frac{-16K(-1)^{m+n}}{\pi^2 (2m+1)(2n+1) \{ \bar{\epsilon}_z \left[\frac{(2m+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2n+1)\pi}{w} \right]^2 \}} \quad (\text{IV-54})$$

The solution for $u''(y,z)$ as a function of the pressure gradient is put in its final form by utilizing the Reynolds analogy and introducing the mixing times T_{cz} , T_{cy} , and T'_c where in this problem T'_c can only be written as

$$T'_c = \frac{T_{cz}}{T_{cy}} \quad (\text{IV-55})$$

since there is no period of oscillation. The resulting expression is

$$u''(y,z) = -\frac{16KT_{cz}}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1)R_{mn}} \left\{ \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} - g(m,n) \right\} \quad (\text{IV-56})$$

where,

$$R_{mn} = \left\{ \left[\frac{(2m+1)\pi}{2} \right]^2 + T'_c \left[\frac{(2n+1)\pi}{2} \right]^2 \right\} \quad (\text{IV-57})$$

To express u'' as a function of $u_{\max}$, (IV-52) is evaluated at $y = z = 0$ and the result solved for K to yield

$$K = -\frac{u_{\max} \pi^2}{16 T_{cz} \phi'} \quad (\text{IV-58})$$

where,

$$\phi' = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1)R_{mn}} \quad (\text{IV-59})$$

Substituting (IV-58) into (IV-56) yields

$$u''(y,z) = \frac{u_{\max}}{\phi'} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1)R_{mn}} \left\{ \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} - g(m,n) \right\} \quad (\text{IV-60})$$

IV-B.2 Concentration Distribution

The boundary value problem for the steady state solution of $c''(y,z)$ using a no flux condition on the perimeter of the section is

$$\text{PDE: } \bar{K}_z \frac{\partial^2 c''}{\partial z^2} + \bar{K}_y \frac{\partial^2 c''}{\partial y^2} = u'' \frac{\partial \bar{C}}{\partial \xi} \quad (\text{IV-61})$$

$$\text{BC's: } \left. \frac{\partial c''}{\partial z} \right|_{z = \pm h} = 0$$

$$\left. \frac{\partial c''}{\partial y} \right|_{y = \pm w/2} = 0$$

Then assuming a solution to (IV-61) of the form

$$c''(y,z) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} a'_{\ell k} \cos \frac{\ell \pi z}{2h} \cos \frac{2k \pi y}{w} \quad (\text{IV-62})$$

and substituting into (IV-61), the result is

$$\begin{aligned} & \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \{ \bar{K}_z \left(\frac{\ell \pi}{h} \right)^2 + \bar{K}_y \left(\frac{2k \pi}{w} \right)^2 \} a'_{\ell k} \cos \frac{\ell \pi z}{h} \cos \frac{2k \pi y}{w} \\ &= \left(- \frac{\partial \bar{C}}{\partial \xi} \right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \left\{ \cos \frac{(2m+1) \pi z}{2h} \cos \frac{(2n+1) \pi y}{w} - g(m,n) \right\} \quad (\text{IV-63}) \end{aligned}$$

Equation (IV-63) is of the same form as (IV-22) thereby allowing the solution for $c''(y,z)$ to be written as

$$c''(y,z) = \left(\frac{\partial \bar{C}}{\partial \xi} \right) \frac{256K}{\pi^4} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k} (-1)^{-(\ell+k)} \cos \frac{\ell \pi z}{h} \cos \frac{2k \pi y}{w}}{\delta_3(\ell, k, p, q)} \quad (\text{IV-64})$$

where,

$$\delta_3(\ell, k, p, q) = [(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2] \cdot$$

$$\{\bar{\epsilon}_z \left[\frac{(2p+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2q+1)\pi}{w} \right]^2\} \{\bar{K}_z \left(\frac{\ell\pi}{h} \right)^2 + \bar{K}_y \left(\frac{2k\pi}{w} \right)^2\} \quad (\text{IV-65})$$

$$\mu_{\ell k} = \begin{cases} 1 & k \neq 0, \ell \neq 0 \\ \frac{1}{2} & k = 0 \text{ or } \ell = 0 \\ 0 & k = \ell = 0 \end{cases}$$

To express c'' as a function of T_{cz} and T'_c Reynolds analogy is applied as before and the mixing time scales are introduced so that (IV-64) becomes

$$c''(y, z) = \left(\frac{\partial \bar{c}}{\partial \xi} \right) \frac{256KT^2_{cz}}{\pi^4} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \cdot$$

$$\frac{\mu_{\ell k} (-1)^{-(\ell+k)} \cos \frac{\ell\pi z}{h} \cos \frac{2k\pi y}{w}}{\delta_4(\ell, k, p, q)} \quad (\text{IV-66})$$

where,

$$\delta_4(\ell, k, p, q) = [(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2] R_{\ell k} R_{pq} \quad (\text{IV-67})$$

$$R_{\ell k} = \{(\ell\pi)^2 + T'_c (k\pi)^2\} \quad (\text{IV-68})$$

$$R_{pq} = \left\{ \left[\frac{(2p+1)\pi}{2} \right]^2 + T'_c \left[\frac{(2q+1)\pi}{2} \right]^2 \right\} \quad (\text{IV-69})$$

The concentration distribution is expressed as a function of $u_{\max}$ by applying (IV-58) to (IV-66) to yield

$$c''(y,z) = \left(-\frac{\partial \bar{C}}{\partial \xi}\right) \frac{16u_{\max} T_{cz}}{\pi^2 \phi'} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k}(-1)^{-(\ell+k)} \cos \frac{\ell \pi z}{h} \cos \frac{2k \pi y}{w}}{\delta_u(\ell, k, p, q)} \quad (IV-70)$$

IV-B.3 Dispersive Mass Transport

The dispersive mass transport resulting from a unidirectional flow through the full rectangular section of Figure 11 is given by

$$\dot{m} = \int_{-h}^h \int_{-w/2}^{w/2} u''(y,z) c''(y,z) dy dz \quad (IV-71)$$

Using (IV-56) for $u''(y,z)$ and (IV-66) for $c''(y,z)$ the integrations in (IV-71) are carried out in the same manner as described in Appendix B. Then, as discussed in IV-A the open channel dispersive mass transport equals one-half $\dot{m}$ and is described as a Fickian flux by

$$-E_L (wh) \frac{\partial \bar{C}}{\partial \xi} = \frac{1}{2} \dot{m} \quad (IV-72)$$

The resulting expression for E_L as a function of the pressure gradient is

$$E_L = \frac{16,384 K^2 T_{cz}^3}{\pi^8} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k}}{\delta_5(\ell, k, m, n, p, q)} \quad (IV-73)$$

where,

$$\delta_5(\ell, k, m, n, p, q) = [(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2] \cdot [(2m+1)^2 - (2\ell)^2][(2n+1)^2 - (2k)^2] R_{mn} R_{pq} R_{\ell k} \quad (IV-74)$$

The longitudinal dispersion coefficient is expressed as a function of $u_{\max}$ by substituting for K in (IV-73) using (IV-58). This expression is put in non-dimensional form by defining

$$E'_L = \frac{E_L}{u_{\max}^2 T_{CZ}} \quad (\text{IV-75})$$

The final form of the solution is summarized below.

$$E'_L = \frac{E_L}{u_{\max}^2 T_{CZ}} = \frac{64}{\pi^4 \phi'^2} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k}}{\delta_s(\ell, k, m, n, p, q)} \quad (\text{IV-76})$$

where,

$$\delta_s(\ell, k, m, n, p, q) = [(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2][(2m+1)^2 - (2\ell)^2] \cdot$$

$$[(2n+1)^2 - (2k)^2] R_{mn} R_{pq} R_{\ell k}$$

$$R_{mn} = \left[\frac{(2m+1)\pi}{2} \right]^2 + T'_C \left[\frac{(2n+1)\pi}{2} \right]^2$$

$$R_{pq} = R_{mn} \text{ with } p \text{ replacing } m \text{ and } q \text{ replacing } n$$

$$R_{\ell k} = [(\ell\pi)^2 + T'_C (k\pi)^2]$$

$$\phi' = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{(2m+1)(2n+1)R_{mn}}$$

$$\mu_{\ell k} = \begin{cases} 1 & k \neq 0, \ell \neq 0 \\ \frac{1}{2} & k = 0, \text{ or } \ell = 0 \\ 0 & k = \ell = 0 \end{cases}$$

It should be noted that the above solution is symmetrical about $T'_C = 1$ in exactly the same manner as the three dimensional oscillatory shear flow solution, and although the discussion to follow is restricted to channels whose half-widths are equal to or greater than their depths the same remarks apply to those channels whose geometries are skewed in the opposite direction.

Equation (IV-76) is plotted in Figure 18 vs. T'_C . In performing the computations a summation limit of 12 was used on each of the six infinite summations given by (IV-76). By adjusting this value in a manner similar to that discussed for the oscillatory case it was concluded that the solution as presented in Figure 18 is very near that given by the infinite summations.

It is seen from Figure 18 that for a unidirectional shear flow in a rectangular channel of finite width the longitudinal dispersion coefficient for wide channels becomes very large. By comparison, if the definition of E'_L given by (IV-75) is applied to the two dimensional solution for E_L given by (III-18) the result is

$$E'_L = \frac{8}{945} = \text{constant} \quad (\text{IV-77})$$

which is shown in Figure 18 by the dotted line. The difference between the behavior of the two solutions is dramatic and for unidirectional flows supports Fischer's theory (6) regarding the dominant effect of lateral shear on longitudinal dispersion in wide channels. The fact that the three dimensional solution does not approach the two dimensional solution in the limit as $T'_C \rightarrow 0$ is not surprising if one realizes that the two problems are completely separate and independent of one another. The three dimensional problem has by definition a lateral shear effect

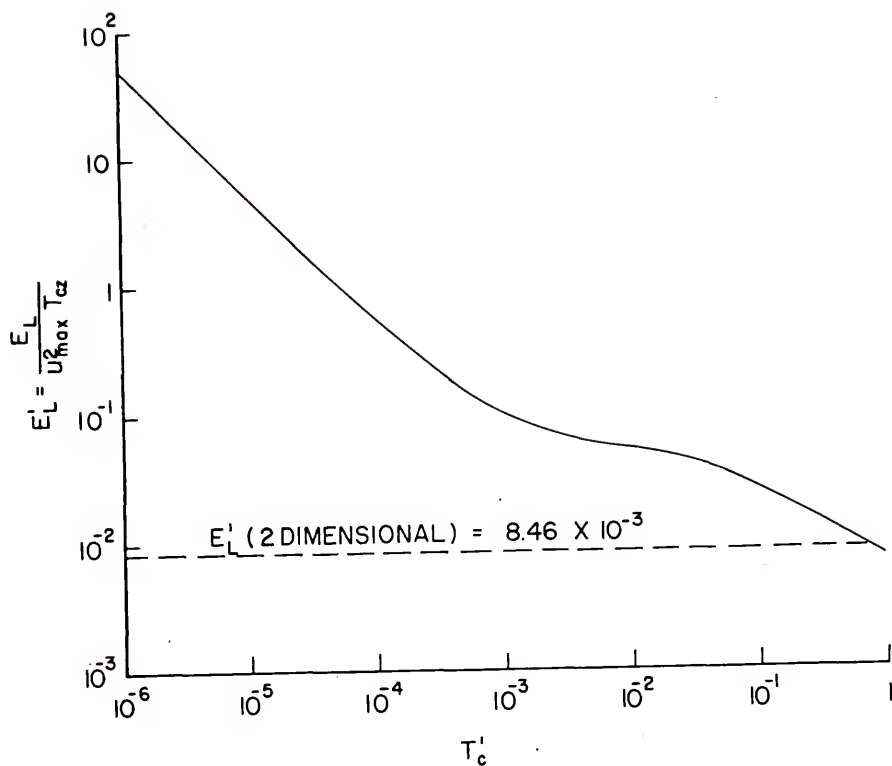


FIGURE 18 NON-DIMENSIONAL LONGITUDINAL DISPERSION COEFFICIENT E'_L , AS A FUNCTION OF T'_c FOR THREE DIMENSIONAL UNIDIRECTIONAL SHEAR FLOW

which no matter how wide the channel, is forever present. Conversely, the two dimensional problem by definition has no such effect. The presence of this shear coupled with a unidirectional flow and a long time scale of lateral mixing must produce a significantly higher longitudinal dispersive mass transport than would be produced in the case where the lateral effect is non-existent.

CHAPTER V
COMPARISON OF RESULTS WITH FIELD AND
LABORATORY DATA

V-A Unidirectional Flow

As an evaluation of the predictive reliability of the solution for the longitudinal dispersion coefficient, E_L , in three dimensional unidirectional shear flow, (IV-76) is compared with field and laboratory data as analyzed by six investigators: Fischer (6, 8, 13), Miller and Richardson (24), Godfrey and Frederick (25), and Thackston and Krenkel (26). As shown in Figure 19 the data are very scattered, and based on first impressions it can only be said that the solution correctly predicts an increasing E_L' with a decreasing T_C' . However, the assumptions made in plotting the data are many and the conditions for which they were obtained, particularly in the field, vary greatly from the assumed straight rectangular geometry used in the analytical solution. Values for the longitudinal dispersion coefficient cannot be measured directly and are obtained only after considerable reduction of data which necessarily involves subjective judgments. This fact is illustrated by comparing the observed E_L' values in Figure 19 as determined by Fischer (8), Godfrey and Frederick (25), and Thackston and Krenkel (26). All of these values were reduced from the same base data.

Perhaps the most limiting qualification of all these data is that none include measured values of both the vertical and lateral eddy diffusivities. Only one set of data, that of Miller and Richardson (24), includes measured values of either K_z or K_y . For their experiment they chose to measure the lateral eddy diffusivity, K_y . The lack of measured K_y and K_z values is unfortunate since it forces one to assume values for T_{cz} , T_{cy} , and T_C' .

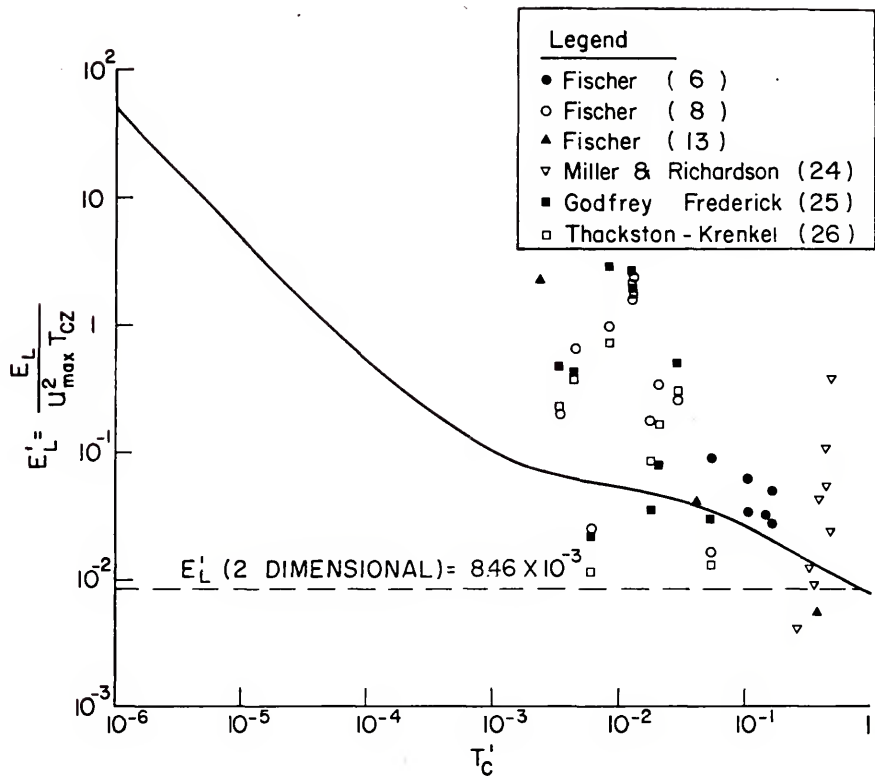


FIGURE 19 COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL DISPERSION IN THREE DIMENSIONAL UNIDIRECTIONAL SHEAR FLOW

Values of $u_{\max}$, the maximum cross-sectional velocity, are also needed for the comparison. Three of the six data sets, Fischer (6, 8), and Miller and Richardson (24), provide cross-sectional mean velocity data for each experimental run and a representative isovel distribution for one run. Using this information it was determined that the relationship

$$u_{\max} = 3/2 \bar{u} \quad (V-1)$$

predicted by the theory used in the present work was in reasonable agreement with the measured data. Additional assumptions were used to obtain $u_{\max}$ values for the Fischer (13) data and these will be discussed separately.

Thus, noting the above considerations, the following assumptions were made in plotting the data shown in Figure 19:

- (1) $\bar{K}_z = 0.063 u_* h$
- (2) $\bar{K}_y = 0.23 u_* h$ (except where measured)
- (3) $u_{\max} = 3/2 \bar{u}$

The expressions used to estimate $\bar{K}_z$ and $\bar{K}_y$ are based on the results of Elder (4). A brief description of the data follows.

The closed circles in Figure 19 represent data obtained by Fischer (6) in a detailed set of three dimensional shear flow laboratory experiments. The flow cross section was trapezoidal with a smooth bottom and rough sides achieved by carefully stacking an embankment of 1 inch stone along the sides of the flume. Flow depths ranged from 2.1 to 4.7 cm while half-widths ranged from 9.5 to 21.5 cm. Measured values of E_L used to compute E_L' are those given by Fischer for Series 2800 through 3200 and 3400 runs using the diffusive transport method. The degree of correlation between these data and the predicted values of E_L' using

(IV-76) is surprisingly good considering the assumptions made and the trapezoidal cross section of the flume.

The open circle data in Figure 19 were reported by Fischer (8) who analyzed field data collected by the U. S. Geological Survey. A detailed description of this set of experiments conducted in several natural streams is given by Godfrey and Frederick (25) whose results are shown in Figure 19 by the closed squares. Thackston and Krenkel (26) also used these data to calculate observed values of E_L which are shown by the open squares in Figure 19. These three sets of data are widely scattered with most points falling above the curve predicted by (IV-76). The general lack of agreement between predicted and observed values of E_L' for these data is not surprising in view of the assumptions made to determine the lateral and vertical mixing times and the non-uniform nature of the natural channels. The largest stream included in the data is the Clinch River, Tennessee having a discharge of 300 cfs and a width-to-depth ratio of nearly 60.

Fischer (13) also reports several experimentally determined values of E_L as determined by Thomas (27), Fischer (28), and Yotsukura et al. (29) for the Chicago Ship Canal, Green-Duwamish River, and Missouri River respectively. These values are plotted in Figure 19 with closed triangles. To obtain values of u_{max} for these data it was assumed that the cross-sectional mean velocity and the shear velocity are related by

$$u_* = \sqrt{\frac{f_{DW}}{8}} \bar{u} \quad (V-2)$$

where f_{DW} is the Darcy-Weisbach friction factor. Then by using a value of 0.08 for f_{DW} and the previously assumed relationship between $\bar{u}$ and u_{max} , the result is

$$u_{\max} = 15 u_* \quad (V-3)$$

The largest channel included in these data is the Missouri River having a width-to-depth ratio of 74.

The final set of data shown in Figure 19 is from a series of laboratory experiments conducted by Miller and Richardson (24). These data are shown by open triangles in the figure and all fall along a narrow band of large T'_C values. The experiments were conducted in a 2 ft wide by 60 ft long recirculating flume with a width-to-depth ratio of approximately 4.7. Turbulent flow conditions were achieved by placing artificial roughness elements at the bed. Roughness element heights ranged in three sizes from 0.43 cm to 2.86 cm compared to a nearly constant flow depth of approximately 13 cm. Cross-sectional mean velocities ranged from 30.5 to 77.4 cm/sec with the higher velocity runs using the largest roughness elements. Experimentally determined values of $\bar{K}_y$ were reported by the authors and were used here to determine the lateral mixing time. The scatter of the Miller and Richardson data is considerable. However, the data for Runs 8 and 9 suggest that the conditions for which a longitudinal dispersion coefficient may be defined were exceeded. Mean velocities for these two runs were 53.3 and 77.4 cm/sec respectively with flow depths of 12.6 and 13.1 cm, and roughness element heights of 2.86 cm. Under these conditions the flow would begin to lose its orderly shear profile characteristics and would certainly exhibit longitudinal non-uniformities due to the relative size of the roughness elements compared to the depth of flow. The authors cite this factor in addition to the entrapment of tracer behind the roughness elements and its subsequent release to the main stream as possible reasons for the abnormally high

values of E_L . In either case it seems reasonable to place a reduced emphasis on the data from Runs 8 and 9 which correspond to the two top open triangles in Figure 19.

None of the experimental data shown correlate particularly well with the analytical solution obtained in Chapter IV for the dispersion coefficient. However, the data are widely scattered with no well defined trend except for a generally increasing E_L' with decreasing T_C' . The importance of measuring K_z and K_y values concurrent with E_L data is definitely emphasized by this behavior. In view of the idealized nature of the solution and the assumptions necessary for a comparison to be made between it and the experimental data, it is felt that the results are reasonable. Certainly, (IV-76) provides a general, predictive expression for the longitudinal dispersion coefficient as a function of both vertical and lateral shear and diffusive effects.

V-B Oscillatory Flow

The comparison of the predicted two and three dimensional oscillatory shear flow dispersion coefficients with field and laboratory data is also difficult due to a general lack of adequate data. Because of the sharply resonant behavior of longitudinal dispersion in oscillatory flow it is essential that measured values for K_z and, where applicable, K_y be obtained concurrently with the experimental determination of $\langle E_x \rangle_T$. Fortunately, this has been done by Bowden (5) for the Mersey Estuary, Liverpool Bay, and a region off the Cumberland Coast, and also by Bowles et al. (30) for a portion of the English Channel off the Dorset Coast.

Three of these field locations may be regarded as two dimensional oscillatory shear flow situations while the fourth, being the Mersey Narrows, must be treated as a three dimensional oscillatory flow. Observed data for the three two dimensional cases are summarized in Table 3.

TABLE 3 OBSERVED DISPERSION DATA AND PREDICTED $\langle E_x \rangle_T$ VALUES USING (III-59)

Parameters	LOCATION		
	Liverpool Bay	Cumberland Coast	English Channel
$\bar{h}$	16 M	24.5 M	28.5 M
T	44,712 sec	44,712 sec	44,712 sec
$u_{\max}$	0.59 M/sec	0.4 M/sec	-
$\langle \bar{u} \rangle_{T/2}$	-	-	0.51 M/sec
$\langle K_z \rangle_T$	0.003 M ² /sec	0.0006 M ² /sec	-
T_{cz}	-	-	1477 sec
$\langle E_x \rangle_T$ (observed)	50 M ² /sec	50 M ² /sec	18 M ² /sec
$\langle E_x \rangle_T$ (predicted)	50.4 M ² /sec	8.6 M ² /sec	9.3 M ² /sec

The values of $\langle E_x \rangle_T$ listed for Liverpool Bay and the Cumberland Coast were determined by Bowden from velocity and salinity profiles taken at one hour intervals throughout the tidal period. From these he was able to determine the dispersive mass transport $\overline{\langle u'' c'' \rangle_T}$. This was then compared with the horizontal salinity gradient through a relationship such as that given by (III-54) to yield a value for $\langle E_x \rangle_T$. Bowden's reported values of $\langle K_z \rangle_T$ were determined by assuming that the vertical distribution of salinity, averaged over a tidal period, was in a steady state condition in the oscillatory sense due to the combined effects of horizontal advection and vertical diffusion. Then using his observed values of u and c he was able to compute $\langle K_z \rangle_T$. Bowden's results show that in each case $\langle K_z \rangle_T$ varies approximately parabolically with depth, being a minimum near the surface, reaching a maximum near mid-depth, and decreasing toward the bottom. It has been assumed here, as previously stated, that this parabolic distribution may be effectively replaced by a constant value corresponding to its mean over depth.

Bowles et al. (30) made no attempt to obtain experimental values for K_z . However, they did provide suitable data for the determination of T_{CZ} directly. In a series of 8 experimental runs in which dye was released as an instantaneous source at the sea bed they experimentally determined the distance downstream at which the dye was considered to be mixed over the water column. Thus, knowing the depth mean velocity one can estimate the vertical mixing time. Values of u_{max} were obtained from $\langle \bar{u} \rangle_{T/2}$ using the temporal and spatial adjustment presented in (V-5) on page 111.

The comparison of results between the observed and predicted values for $\langle E_X \rangle_T$ is illustrated by Figure 20. The observed value of $\langle E_X \rangle_T$ for the Cumberland Coast corresponds to $E_X^1 = 7 \times 10^{-3}$ which plots off-scale and is indicated by an arrow. As shown in Table 3 this value is slightly less than 6 times the predicted value. However, this is not surprising if one considers the physical difference between the Cumberland Coast and an infinitely wide rectangular channel. In fact, the agreement between the predicted and observed values of $\langle E_X \rangle_T$ for the remaining two locations is remarkable. For Liverpool Bay the two values are essentially the same, whereas for the more open English Channel they agree within a factor of 2.

The most comprehensive set of data available for evaluating values of $\langle E_X \rangle_T$ predicted using the methods presented here is given by Bowden (31, 5), and Fischer (12) for the Mersey Estuary Narrows. The Mersey Narrows has a width-to-depth ratio of approximately 67 and a T_Z^1 value of 4.25. Thus, from Table 2 it is seen that this case must be treated as a three dimensional oscillatory shear flow. In reporting his data Bowden notes the significant effect of vertical density stratification on longitudinal dispersion. Observed values of the vertical eddy diffusivity are one to two orders of magnitude below the expected value for a comparable situation in a homogeneous fluid. Bowden attributes this to an observed top to bottom salinity difference of 1.3 to 5.0 parts per thousand and concludes that the small values of $\langle \bar{K}_Z \rangle_T$ contribute significantly to the large observed values of $\langle E_X \rangle_T$. Table 4 summarizes the observed data and gives the predicted value of $\langle E_X \rangle_T$ obtained from (IV-43). The value of $\bar{K}_y$ given in Table 4 was determined by using Ward's (23) experimental result that

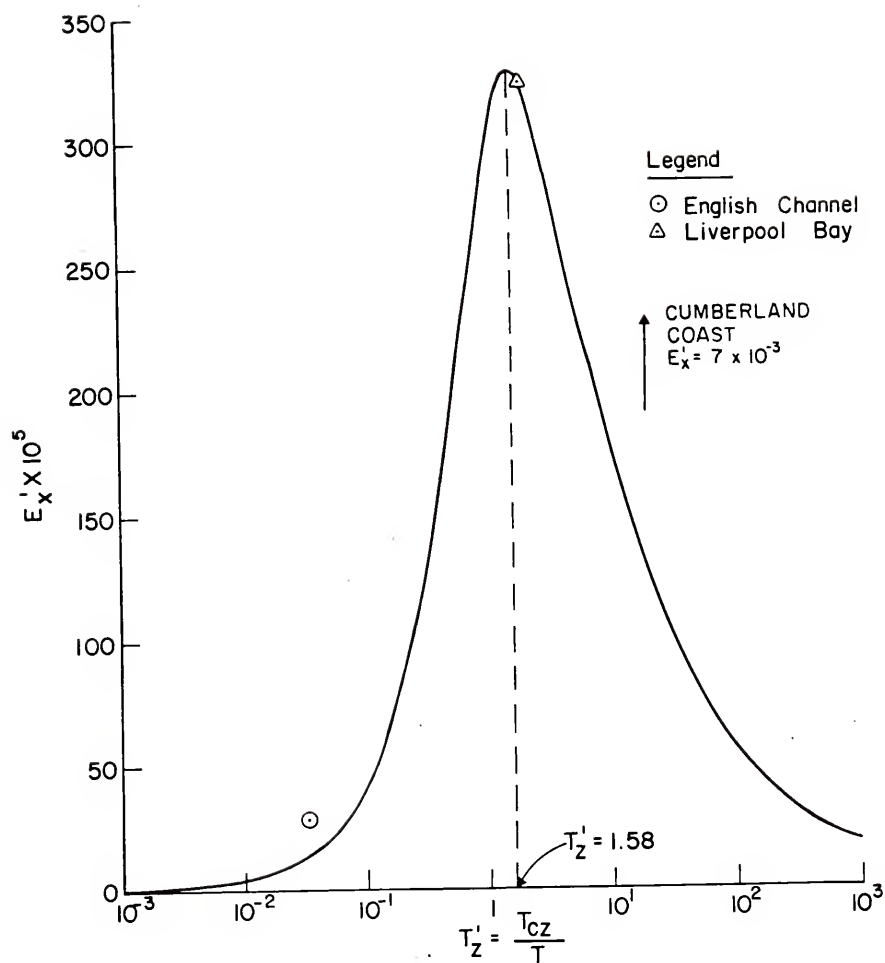


FIGURE 20 COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL DISPERSION FOR THE ENGLISH CHANNEL, LIVERPOOL BAY, AND CUMBERLAND COAST

$$\bar{k}_y = 0.45 u_* h \quad (V-4)$$

TABLE 4 OBSERVED DISPERSION DATA AND PREDICTED $\langle E_x \rangle_T$ VALUES FOR THE MERSEY NARROWS

Parameter	Value
$\bar{h}$	19.5 M
w	1300 M
u_{max}	1.41 M/sec
T	44,712 sec
u_*	0.04 M/sec
$\langle R_z \rangle_T$	0.002 M ² /sec
$\langle \bar{k}_y \rangle_T$, (Eq. (V-4))	0.35 M ² /sec
T_{cz}	190,100 sec
T_{cy}	1,203,700 sec
T'_z	4.25
T'_c	0.16
Range of Observed $\langle E_x \rangle_T$	161 to 360 M ² /sec
$\langle E_x \rangle_T$ using $\overline{\langle u'' c'' \rangle_T}$ data	170 M ² /sec
$\langle E_x \rangle_T$ using salt balance	265 M ² /sec
$\langle E_x \rangle_T$ predicted by Eq. (IV-43)	259 M ² /sec

Several values of $\langle E_x \rangle_T$ based upon observed data were reported by Bowden all of which fall within a range of 161 to 360 M²/sec. Bowden (5) obtained values of 170 and 265 M²/sec using measured $\overline{\langle u'' c'' \rangle_T}$ data as discussed previously and a steady state salinity analysis respectively. These two values for $\langle E_x \rangle_T$ are listed separately in Table 4 in addition to the observed range for $\langle E_x \rangle_T$.

Figure 21 shows the observed range for $\langle E_x \rangle_T$ and Bowden's two values obtained from observed data as they compare to the predicted resonance curve for E'_x in the Mersey Narrows. The predicted curve passes almost exactly through the mid-point of the observed range of E'_x values and is within 3 per cent of Bowden's observed E'_x value using the salt balance analysis. However, the predicted E'_x is 52 per cent greater than Bowden's observed E'_x using $\langle \bar{u}''c'' \rangle_T$. In general, the agreement is considered to be good.

All of the longitudinal dispersion data discussed thus far for oscillatory flows have been obtained in the field. There are surprisingly few good data obtained in the laboratory for open channel oscillatory flow. Fukuoka (32) has conducted an extensive set of experiments designed to look at this problem but unfortunately he made no attempt to measure K_z or K_y . Segall (19) also conducted a series of laboratory experiments to study longitudinal dispersion in oscillatory open channel flow in which he experimentally determined a mean value for K_z . A summary of Segall's data along with the predicted value for $\langle E_x \rangle_T$ using (IV-43) is presented in Table 5. To determine a value for T_{cy} , the lateral eddy diffusivity was estimated by using (V-2) and (V-4), where $\bar{u}$ is replaced by $\langle \bar{u} \rangle_{T/2}$ and it is assumed that

$$\langle \bar{u} \rangle_{T/2} = \frac{4}{3\pi} u_{\max} \quad (V-5)$$

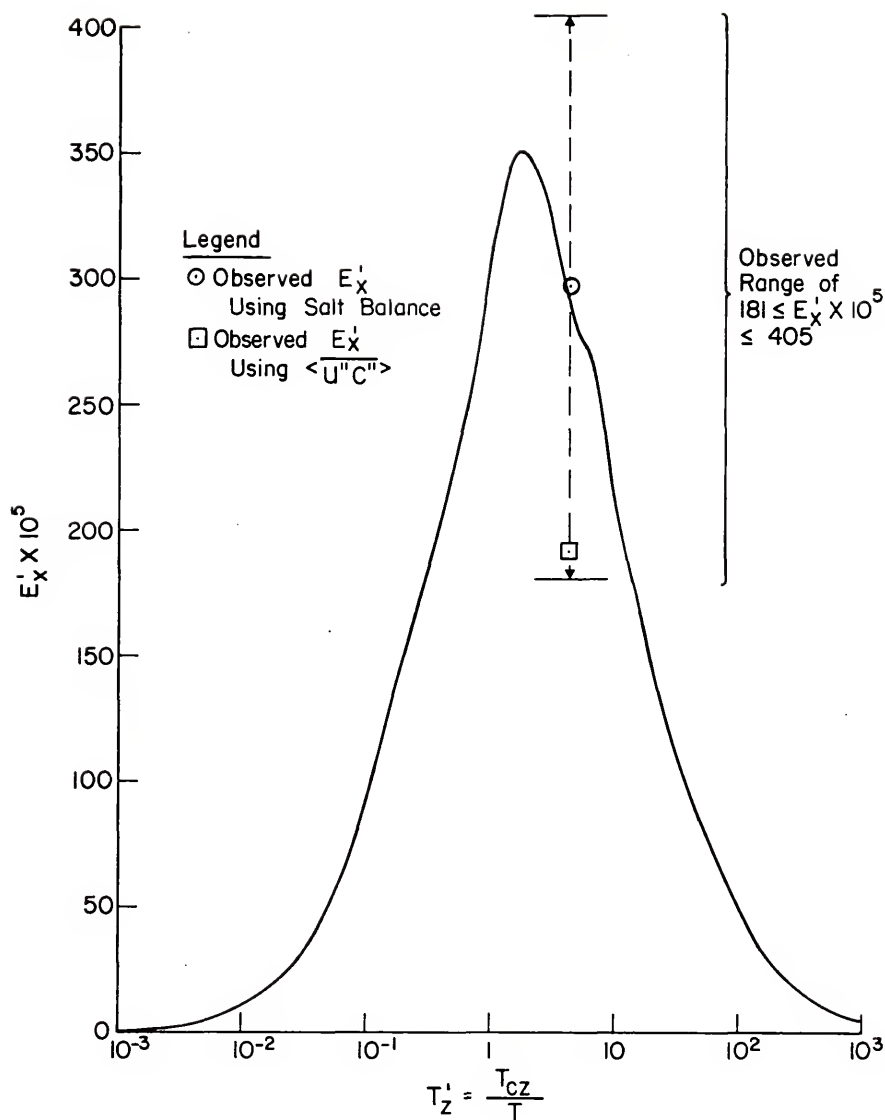


FIGURE 21 COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL DISPERSION FOR THE MERSEY ESTUARY

TABLE 5 OBSERVED DISPERSION DATA AS REPORTED
BY SEGALL AND CORRESPONDING PREDICTED
 $\langle E_x \rangle_T$ VALUES

Parameter	Value
$\bar{h}$	0.44 ft
w	3.0 ft
T	437 sec
u_{max}	0.1 ft/sec
$\langle \bar{K}_z \rangle_T$	0.001 ft ² /sec
$\langle \bar{K}_y \rangle_T$	0.00084 ft ² /sec
T_{cz}	193.6 sec
T_{cy}	2678.6 sec
T'_c	0.07
$\langle E_x \rangle_T$ observed	0.013 ft ² /sec
$\langle E_x \rangle_T$ predicted using (IV-43)	0.009 ft ² /sec

Figure 22 shows the E'_x resonance curve for $T'_c = 0.07$ corresponding to Segall's data with his measured value of E'_x plotted. It is seen that the predicted value for E'_x is approximately 70 per cent of the observed value. It should be noted, however, that as shown by Figure 22 the predicted value of E'_x is extremely sensitive to T'_c for the conditions used by Segall. Thus, errors made in the determination of $\bar{h}$, $\bar{K}_z$, or T could have a significant effect on the predicted value for $\langle E_x \rangle_T$. Moreover, it appears that Segall's experimental value of 0.013 ft²/sec for $\langle E_x \rangle_T$ was determined prior to the establishment of a true periodic condition for the concentration. As discussed in II-C this condition is required

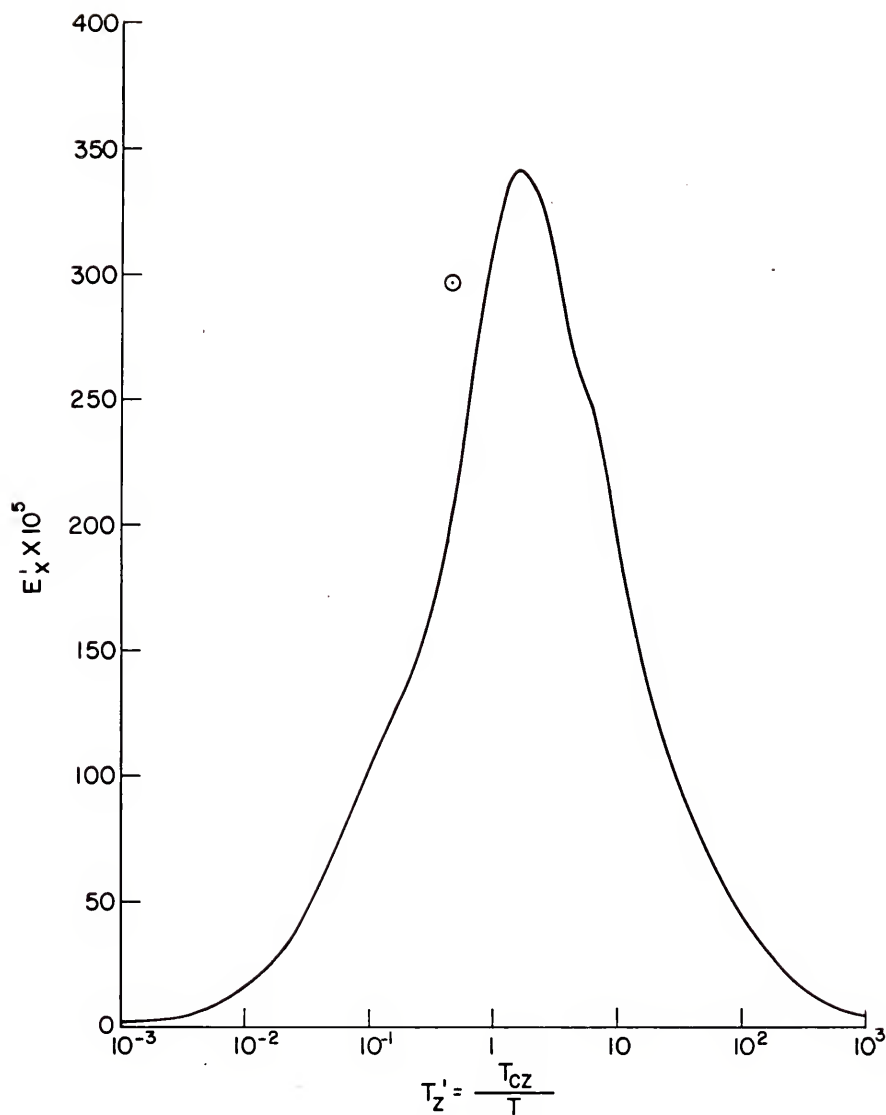


FIGURE 22 COMPARISON OF PREDICTED AND OBSERVED LONGITUDINAL DISPERSION USING SEGALL'S DATA (19)

for the existence of a stationary random dispersive process. This is illustrated by Segall's Figure 6.1 in which it is seen that $\langle E_x \rangle_T$ is still decreasing with time during tidal cycles 2, 3, and 4 which he used to determine his experimental coefficient. However, the rate of decrease becomes small by tidal cycle 4 and if one extrapolates the data shown it is seen that a more correct value of $\langle E_x \rangle_T$ from Segall's experiment would be very close to the predicted value of $0.009 \text{ ft}^2/\text{sec}$ given in Table 5:

The importance of having good estimated values for T_{CZ} , T_{CY} , and T'_C has been demonstrated by the generally good agreement found between observed and predicted $\langle E_x \rangle_T$ values when adequate data exist and the correspondingly poor agreement found between these values when adequate data do not exist. This applies to both unidirectional and oscillatory shear flow systems but is of particular importance to the latter due to its resonant characteristic.

CHAPTER VI SUMMARY AND CONCLUSIONS

Analytical expressions for the cross-sectional variation components of velocity and concentration, u'' and c'' , and the longitudinal dispersion coefficient, E_L and $\langle E_X \rangle_T$, have been developed using Taylor's (1) definition of dispersion for 4 cases of shear flows:

- (1) Unidirectional flow in an infinitely wide rectangular channel,
- (2) Oscillatory flow in an infinitely wide rectangular channel,
- (3) Unidirectional flow in a rectangular channel of finite width,
- (4) Oscillatory flow in a rectangular channel of finite width.

It was shown in Chapter II that Taylor's definition of the longitudinal convective mass transport of a substance with respect to the mean flow is valid in unidirectional flows under the assumptions of steady uniform flow. In oscillatory flows the required assumptions are that the velocity and concentration functions be temporally periodic and that the flow be uniform. When these requirements are satisfied for their respective flow conditions the time averaged convective mass transport of substance with respect to the mean flow, $\langle u''c'' \rangle_T$, is a stationary random process and is therefore Gaussian in behavior. This permits its reformulation as a Fickian flux.

Specific conclusions regarding the predicted functional behavior of longitudinal dispersive mass transport and the comparison of the analytical solutions developed in Chapters III and IV with observed data are:

- (1) Because of the distinctly different hydrodynamic and kinematic characteristics of unidirectional and oscillatory flow systems the

functional behavior of longitudinal dispersive mass transport in each system is unique. This fact is most easily seen for two dimensional shear flows in which the dispersion coefficient for unidirectional flow varies directly with the time of vertical mixing whereas for oscillatory flow the dispersion coefficient is governed by a resonant interaction between the period of oscillation and the time of vertical mixing. It is shown that as the period of oscillation becomes very large the dispersion coefficient approaches one-half the value of the coefficient for a comparable unidirectional flow.

(2) The description of longitudinal dispersion in oscillatory flow through its relationship to a "corresponding" process in unidirectional flow is misleading for reasons stated above. Although technically correct values for $\langle E_x \rangle_T$ may be obtained in this manner, great care should be exercised in the selection of expressions used for $\langle E_x \rangle_T$ and E_L . This choice is by no means arbitrary and ideally should be limited to companion solutions of the steady state and oscillatory boundary value problems of flow and transport diffusion such as those presented in Chapters III and IV.

(3) A solution for the longitudinal dispersion coefficient in two dimensional oscillatory shear flow is developed and presented in the form of a single non-dimensional resonance curve whose maximum occurs when the vertical mixing time is 1.58 times the period of oscillation. This curve may be used to obtain predicted values of the longitudinal dispersion coefficient for channels satisfying the two dimensional criteria discussed in Chapter IV by knowing: (a) the period of oscillation, (b) the surface amplitude of the velocity, (c) the water depth, and (d) the depth mean value of vertical eddy diffusivity.

(4) The longitudinal dispersive mass transport in a rectangular channel of finite width is symmetric about $T'_C = 1$ for both unidirectional and oscillatory flows. Thus, a channel whose half-width was n times its depth would exhibit the same longitudinal dispersion as a channel whose depth was n times its half-width provided that:

- (a) $u_{\max}$ is the same for both cases,
- (b) The vertical and lateral mixing times for the first case were equal respectively to the lateral and vertical mixing times for the second case,
- (c) The periods of oscillation for oscillatory flows were the same.

(5) The additional feature of a finite channel width in oscillatory shear flow can either increase or decrease the longitudinal dispersion. The sense of this effect is determined by the relative magnitudes of the vertical mixing time, lateral mixing time, and period of oscillation. For $T'_C = 1$ the peak of the resonance curve for $\langle E_x \rangle_T$ occurs at T'_Z (or symmetrically T'_Y) = 4.5. As T'_C approaches zero the peak first increases in magnitude reaching a maximum for $T'_C = 0.25$ and then decreases to the limiting value for the two dimensional oscillatory shear case. At this point the peak occurs at $T'_Z = 1.58$ and the resonance curve for $\langle E_x \rangle_T$ is identical to the one obtained for two dimensional oscillatory flow. Predictive values for the longitudinal dispersion coefficient are obtainable from these curves by knowing: (a) channel width, (b) channel depth, (c) centerline surface amplitude of velocity, (d) period of oscillation, and (e) cross-sectional mean values of the vertical and lateral coefficients of eddy diffusivity.

(6) Longitudinal dispersive mass transport in finite width rectangular channels with unidirectional shear flow does not approach the two dimensional solution in the limit as one of the channel dimensions becomes very large. In fact, E_L becomes infinitely large as the channel geometry is skewed in either depth or width. This supports Fischer's theory regarding the dominant effect of lateral shear on longitudinal dispersion in natural channels where the width-to-depth ratio is much larger than unity.

(7) A comparison has been made between predicted and observed values of the longitudinal dispersion coefficient using solutions developed in Chapters III and IV and the field and laboratory data of several investigators. These comparisons demonstrate the necessity for good estimates of the vertical and lateral mixing times as part of all longitudinal dispersion experiments. Where these data are available the agreement between the predicted and observed coefficients is very good. However, where values of T_{cz} and T_{cy} must be estimated the data become very scattered. This is particularly apparent for the three dimensional unidirectional shear flow case for which nearly all T_{cz} and T_{cy} values had to be estimated.

APPENDIX A

DERIVATION OF NORMALIZING EXPRESSIONS FOR TWO DIMENSIONAL OSCILLATORY SHEAR FLOW SOLUTION

A-1 Surface Amplitude of Velocity, $u_{\max}$

To obtain an expression for $u_{\max}$ as a function of the pressure gradient modulus, K , the solution for $u(z,t)$ is first evaluated at the free surface and then maximized with respect to time. The desired expression for $u(z,t)$ is given by Equation (III-23) as

$$u(z,t) = \frac{iK}{\sigma} \left[1 - \frac{\cosh \beta(1+i)(h-z)}{\cosh \beta(1+i)h} \right] e^{i\sigma t} \quad (A-1)$$

where,

$$\beta = \sqrt{\frac{\sigma}{2\epsilon_z}} \quad (A-2)$$

To begin, evaluate (A-1) at $z = h$,

$$u(h,t) = \frac{iK}{\sigma} e^{i\sigma t} \left[1 - \frac{1}{\cosh \beta(1+i)h} \right] \quad (A-3)$$

where,

$$\cosh \beta(1+i)h = \frac{1}{\sqrt{2}} (\cosh 2\beta h + \cos 2\beta h)^{\frac{1}{2}} e^{i\alpha_2} \quad (A-4)$$

and,

$$\alpha_2 = \tan^{-1} \left(\frac{\sinh \beta h \sin \beta h}{\cosh \beta h \cos \beta h} \right) \quad (A-5)$$

Next, substitute (A-4) into (A-3) and take the real part of the resulting expression to yield

$$R_e[u(h,t)] = -\frac{K}{\sigma} \sin \sigma t + \frac{\sqrt{2} K \sin (\sigma t - \alpha_2)}{\sigma [\cosh 2\beta h + \cos 2\beta h]^{\frac{1}{2}}} \quad (\text{A-6})$$

The surface amplitude of the velocity is given by (A-6) evaluated at time $t = t_{\max}$, i.e. the time at which the maximum velocity, $u_{\max}$, occurs. To determine this, (A-6) is differentiated with respect to time and set equal to zero. Thus,

$$-K \cos \sigma t_{\max} + \frac{\sqrt{2} K (\cos \sigma t_{\max} \cos \alpha_2 + \sin \sigma t_{\max} \sin \alpha_2)}{(\cosh 2\beta h + \cos 2\beta h)^{\frac{1}{2}}} = 0$$

Solving for $\sigma t_{\max}$,

$$\sigma t_{\max} = \tan^{-1} \left\{ \frac{(\cosh 2\beta h + \cos 2\beta h)^{\frac{1}{2}} - \sqrt{2} \cos \alpha_2}{\sqrt{2} \sin \alpha_2} \right\} \quad (\text{A-7})$$

Equation (A-7) is then substituted into (A-6) for σt to obtain the desired expression for $u_{\max}$. To simplify the algebra, let

$$F = \frac{1}{\sqrt{2}} (\cosh 2\beta h + \cos 2\beta h)^{\frac{1}{2}} \quad (\text{A-8})$$

Then,

$$\sin \sigma t_{\max} = \frac{F - \cos \alpha_2}{(F^2 - 2F \cos \alpha_2 + 1)^{\frac{1}{2}}} \quad (\text{A-9})$$

and,

$$\cos \sigma t_{\max} = \frac{\sin \alpha_2}{(F^2 - 2F \cos \alpha_2 + 1)^{\frac{1}{2}}} \quad (\text{A-10})$$

Upon substitution of (A-8) through (A-10), Equation (A-6) becomes

$$u_{\max} = \frac{K}{\sigma F (F^2 - 2F \cos \alpha_2 + 1)^{\frac{1}{2}}} \left\{ F \cos \alpha_2 - F^2 + F \cos \alpha_2 - \cos^2 \alpha_2 - \sin^2 \alpha_2 \right\}$$

or,

$$u_{\max} = - \frac{K}{\sigma F} (F^2 - 2F \cos \alpha_2 + 1)^{\frac{1}{2}} \quad (\text{A-11})$$

Now, from (A-5),

$$\cos \alpha_2 = \frac{\cosh \beta h \cos \beta h}{F} \quad (\text{A-12})$$

Substituting (A-12) into (A-11) and solving for K yields,

$$K = \frac{-\sigma F u_{\max}}{[\cosh 2\beta h + \cos 2\beta h - 2 \cosh \beta h \cos \beta h + 1]^{\frac{1}{2}}}$$

which, after some algebra and the substitution for F using (A-8) yields as a result,

$$K = - \frac{\sigma u_{\max} (\cosh 2\beta h + \cos 2\beta h)^{\frac{1}{2}}}{\sqrt{2} (\cosh \beta h - \cos \beta h)} \quad (\text{A-13})$$

Equation (A-13) is seen to be identical to (III-41) by substituting $\frac{2\pi}{T}$ for σ and applying (III-36) which states

$$\beta h = \sqrt{\pi T \frac{1}{Z}}$$

then,

$$K = - \frac{\sqrt{2} \pi u_{\max} (\cosh 2 \sqrt{\pi T'_Z} + \cos 2 \sqrt{\pi T'_Z})^{\frac{1}{2}}}{T (\cosh \sqrt{\pi T'_Z} - \cos \sqrt{\pi T'_Z})} \quad (\text{A-14})$$

This is the expression used in Chapter III to relate the pressure gradient modulus, K , to the surface amplitude of velocity, $u_{\max}$.

A-2 Surface Excursion Length, L

The expression of the pressure gradient modulus as a function of the surface excursion length is simplified through the use of (A-14). This is seen by first noting that the surface excursion length is defined as the product of the time mean surface velocity during one-half of a period of oscillation and the time of flow, or

$$L = \langle u(h,t) \rangle_{T/2} \cdot \frac{T}{2} \quad (\text{A-15})$$

However, the surface velocity amplitude, $u_{\max}$, is given by

$$u_{\max} = \frac{\pi}{2} \langle u(h,t) \rangle_{T/2} \quad (\text{A-16})$$

Thus,

$$L = \frac{T}{\pi} u_{\max} \quad (\text{A-17})$$

Combining (A-17) with (A-14) yields the desired relationship between K and L ,

$$K = - \frac{\sqrt{2} \pi^2 L (\cosh 2 \sqrt{\pi T'_Z} + \cos 2 \sqrt{\pi T'_Z})^{\frac{1}{2}}}{T^2 (\cosh \sqrt{\pi T'_Z} - \cos \sqrt{\pi T'_Z})} \quad (\text{A-18})$$

APPENDIX B

MATHEMATICAL DETAILS FOR THREE DIMENSIONAL OSCILLATORY SHEAR FLOW DISPERSION

B-1 Velocity Solution

The governing partial differential equation for the spatial distribution of velocity over the flow cross section is given by Equation (IV-3) as

$$\bar{\epsilon}_z \frac{\partial^2 u_s}{\partial z^2} + \bar{\epsilon}_y \frac{\partial^2 u_s}{\partial y^2} - i \sigma u_s = K \quad (\text{B-1})$$

for which a solution of the form

$$u_s(y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} \quad (\text{B-2})$$

is assumed.

Substituting (B-2) into (B-1) and performing the implied operations yields,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \{ \bar{\epsilon}_z \left[\frac{(2m+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2n+1)\pi}{w} \right]^2 + i\sigma \} . \quad (\text{B-3})$$

$$a_{mn} \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} = -K$$

Next multiply (B-3) by $\cos \frac{(2\ell + 1)\pi z}{2h}$ and integrate from $-h$ to h , noting that,

$$\int_{-h}^h \cos \frac{(2\ell + 1)\pi z}{2h} \cos \frac{(2m + 1)\pi z}{2h} dz = \begin{cases} 0, & \ell \neq m \\ h, & \ell = m \end{cases}$$

Equation (B-3) is then reduced to a single sum in n and may be written after all integrations have been carried out as

$$\sum_{n=0}^{\infty} \{ \bar{\epsilon}_z \left[\frac{(2\ell + 1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2n+1)\pi}{w} \right]^2 + i\sigma \} .$$

$$a_{\ell n} \cos \frac{(2n+1)\pi y}{w} = - \frac{4K(-1)^\ell}{(2\ell+1)\pi} \quad (\text{B-4})$$

To eliminate the summation in n multiply (B-4) by $\cos \frac{(2k+1)\pi y}{w}$ and integrate from $-w/2$ to $w/2$, where now

$$\int_{-w/2}^{w/2} \cos \frac{(2k+1)\pi y}{w} \cos \frac{(2n+1)\pi y}{w} dy = \begin{cases} 0, & n \neq k \\ w/2, & n = k \end{cases}$$

Equation (B-4) thus becomes

$$\begin{aligned} \{ \bar{\epsilon}_z \left[\frac{(2\ell+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2k+1)\pi}{w} \right]^2 + i\sigma \} a_{\ell k} \\ = - \frac{8K(-1)^\ell}{w(2\ell+1)\pi} \int_{-w/2}^{w/2} \cos \frac{(2k+1)\pi y}{w} dy \end{aligned} \quad (\text{B-5})$$

Carrying out the remaining integration, solving (B-5) for $a_{\ell k}$, and replacing ℓ with m and k with n , the solution for the Fourier coefficient is

$$a_{mn} = - \frac{16 K (-1)^{m+n}}{(2m+1)(2n+1)\pi^2 \{ \bar{\epsilon}_z \left[\frac{(2m+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2n+1)\pi}{w} \right]^2 + i\sigma \}} \quad (\text{B-6})$$

This is seen to coincide with (IV-9). The desired solution for the velocity distribution, $u(y,z,t)$, is then obtained by substituting (B-6) into (B-2) and multiplying the result by $e^{i\sigma t}$, or

$$u(y,z,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{-16 K(-1)^{m+n} \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} e^{i\sigma t}}{(2m+1)(2n+1)\pi^2 \{ \bar{\epsilon}_z \left[\frac{(2m+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2n+1)\pi}{w} \right]^2 + i\sigma \}} \quad (B-7)$$

which completes the required derivation of (IV-5).

B-2 Concentration Solution

The governing equation for the spatial distribution of the concentration variation over the flow cross section is given by (IV-20) as

$$\bar{k}_z \frac{\partial^2 c_s''}{\partial z^2} + \bar{k}_y \frac{\partial^2 c_s''}{\partial y^2} - i\sigma c_s'' = u_s'' \frac{\partial \bar{c}}{\partial \xi} \quad (\text{B-8})$$

where,

$$u_s''(y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \left\{ \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} - g(m, n) \right\} \quad (\text{B-9})$$

a_{mn} is given by (B-6), and

$$g(m, n) = \frac{4(-1)^{m+n}}{\pi^2(2m+1)(2n+1)} \quad (\text{B-10})$$

Equation (B-9) is then used to substitute for u_s'' in (B-8) and a solution for c_s'' of the form

$$c_s''(y, z) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} a'_{\ell k} \cos \frac{\ell\pi z}{h} \cos \frac{2k\pi y}{w} \quad (\text{B-11})$$

is assumed and substituted into (B-8) to yield

$$\sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \left\{ \bar{k}_z \left(\frac{\ell\pi}{h} \right)^2 + \bar{k}_y \left(\frac{2k\pi}{w} \right)^2 + i\sigma \right\} a'_{\ell k} \cos \frac{\ell\pi z}{h} \cos \frac{2k\pi y}{w} =$$

$$\left(- \frac{\partial \bar{c}}{\partial \xi} \right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \left\{ \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} - g(m, n) \right\} \quad (\text{B-12})$$

To solve for $a'_{\ell k}$, (B-12) is first multiplied by $\cos \frac{2j\pi y}{w}$ and integrated from $-w/2$ to $w/2$. On the left hand side of (B-12) the integration is

$$\int_{-w/2}^{w/2} \cos \frac{2j\pi y}{w} \cos \frac{2k\pi y}{w} dy = \begin{cases} 0, & k \neq j \\ w/2, & k = j \end{cases} \quad (\text{B-13})$$

which reduces the left hand side to a single sum in ℓ multiplied by $w/2$. However, the integration over y on the right hand side of (B-12) does not cause a similar reduction in the number of sums because of the non-orthogonality of the cosine terms. This is seen by defining the integral over y by I_1 . Then,

$$\begin{aligned}
 I_1 &= \int_{-w/2}^{w/2} \cos \frac{(2n+1)\pi y}{w} \cos \frac{2j\pi y}{w} dy & (B-14) \\
 &= \int_0^{w/2} \cos (2n - 2j+1) \frac{\pi y}{w} dy + \int_0^{w/2} \cos (2n + 2j+1) \frac{\pi y}{w} dy \\
 &= \frac{w}{\pi} \left\{ \frac{\cos (n-j)\pi}{(2n - 2j+1)} + \frac{\cos (n+j)\pi}{(2n + 2j+1)} \right\} \\
 &= \frac{w}{\pi} \left\{ \frac{(-1)^{n-j}}{(2n - 2j+1)} + \frac{(-1)^{n+j}}{(2n + 2j+1)} \right\}
 \end{aligned}$$

or,

$$I_1 = \frac{2w(-1)^{n-j}}{\pi} \left\{ \frac{2n+1}{(2n+1)^2 - (2j)^2} \right\} \quad (B-15)$$

The $g(m,n)$ term on the right hand side of (B-12) does not survive the integration since

$$\int_{-w/2}^{w/2} g(m,n) \cos \frac{2j\pi y}{w} dy = g(m,n) \int_{-w/2}^{w/2} \cos \frac{2j\pi y}{w} dy = 0$$

Equation (B-12) may then be written as

$$\begin{aligned}
 \sum_{\ell=0}^{\infty} \{ \bar{K}_z \left(\frac{\ell\pi}{h} \right)^2 + \bar{K}_y \left(\frac{2j\pi}{w} \right)^2 + i\sigma \} a'_{\ell j} \cos \frac{\ell\pi z}{h} & \quad (B-16) \\
 = \left(-\frac{\partial \bar{c}}{\partial \xi} \right) \frac{4}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \frac{(-1)^{n-j} (2n+1) \cos \frac{(2n+1)\pi z}{2h}}{[(2n+1)^2 - (2j)^2]}
 \end{aligned}$$

The remaining sum on the right hand side of (B-16) is eliminated in a similar manner, i.e. by multiplying (B-16) by $\cos \frac{f\pi z}{h}$ and integrating from $-h$ to h . From (B-13) the left hand side integral is

$$\int_{-h}^h \cos \frac{f\pi z}{h} \cos \frac{\ell\pi z}{h} dz = \begin{cases} 0, & \ell \neq f \\ h, & \ell = f \end{cases} \quad (\text{B-17})$$

Let I_2 represent the right hand side integral over z . Then,

$$I_2 = \int_{-h}^h \cos \frac{(2m+1)\pi z}{2h} \cos \frac{f\pi z}{h} dz \quad (\text{B-18})$$

and proceeding in a manner similar to that done for I_1 ,

$$I_2 = \frac{4h(-1)^{m-f}}{\pi} \left\{ \frac{(2m+1)}{(2m+1)^2 - (2f)^2} \right\} \quad (\text{B-19})$$

With the aid of (B-17) and (B-19), (B-16) may now be written as,

$$a'_{fj} \left\{ \bar{K}_z \left(\frac{f\pi}{h} \right)^2 + \bar{K}_y \left(\frac{2j\pi}{w} \right)^2 + i\sigma \right\} = \left(-\frac{\partial \bar{C}}{\partial \xi} \right) \frac{16}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \quad (\text{B-20})$$

$$\frac{a_{mn} (-1)^{m+n-f-j} (2m+1)(2n+1)}{[(2m+1)^2 - (2j)^2][(2n+1)^2 - (2f)^2]}$$

The desired solution for $a'_{\ell k}$ is then obtained by solving (B-20) for a'_{fj} and replacing f with ℓ and j with k . In addition, to maintain the general nature of the summation indices, m is replaced with p and n is replaced with q . Thus, the solution for $a'_{\ell k}$ after substituting for a_{pq} using (B-6) is :

$$a'_{\ell k} = \left(\frac{\partial \bar{c}}{\partial \bar{t}} \right) \frac{256 K (-1)^{-(\ell+k)}}{\pi^4 \left\{ \bar{K}_z \left(\frac{\ell \pi}{h} \right)^2 + \bar{K}_y \left(\frac{2k\pi}{w} \right)^2 + i\sigma \right\}} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \quad (B-21)$$

$$\frac{1}{[(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2] \left\{ \bar{\epsilon}_z \left[\frac{(2p+1)\pi}{2h} \right]^2 + \bar{\epsilon}_y \left[\frac{(2q+1)\pi}{w} \right]^2 + i\sigma \right\}}$$

The spatial variation of concentration, $c''(y,z,t)$, is then written as

$$c''(y,z,t) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \mu_{\ell k} a'_{\ell k} \cos \frac{\ell \pi z}{h} \cos \frac{2k \pi y}{w} e^{i\sigma t} \quad (B-22)$$

where,

$$\mu_{\ell k} = \begin{cases} 0, & k = \ell = 0 \\ \frac{1}{2}, & k = 0 \text{ or } \ell = 0 \\ 1, & k \neq 0 \text{ and } \ell \neq 0 \end{cases} \quad (B-23)$$

The factor, $\mu_{\ell k}$, must be included to properly account for the Fourier coefficients a'_{00} , a'_{0k} , and $a'_{\ell 0}$. This is seen by first considering the definition of $c''(y,z,t)$. As expressed by (I-4) and (II-7), the concentration, c , is decomposed into its cross-sectional mean and variation components by

$$c = \bar{c} + c''$$

and thus by definition,

$$\int_{-h}^h \int_{-w/2}^{w/2} c'' dy dz = 0 \quad (B-24)$$

Thus, the $\ell = 0$ and $k = 0$ term in (B-22) must be zero to satisfy (B-24) and thus $\mu_{00} = 0$. For the case where either ℓ or k is zero it is seen from

(B-13) and (B-17) that the surviving term is twice as large as the values shown. Since a'_{lk} varies as the inverse of these terms, the values of μ_{0k} and μ_{l0} must necessarily equal 0.5.

The expression given for $c''(y,z,t)$ by (B-22) is equivalent to that given by (IV-24). To express c'' as given by (B-22) in the same manner as given by (IV-24) it is necessary to: (1) substitute for a'_{lk} using (B-21); (2) introduce the definitions of σ and the vertical and lateral mixing times; and (3) transform all complex factors to polar form. Take, for example, the factor $\{\bar{K}_z(\frac{l\pi}{h})^2 + \bar{K}_y(\frac{2k\pi}{w})^2 + i\sigma\}$ and recall the following definitions:

$$\begin{aligned}\sigma &= \frac{2\pi}{T} \\ T_{cz} &= h^2/\bar{K}_z \\ T_{cy} &= (w/2)^2/\bar{K}_y\end{aligned}$$

Then,

$$\{\bar{K}_z(\frac{l\pi}{h})^2 + \bar{K}_y(\frac{2k\pi}{w})^2 + i\sigma\} = \frac{2\pi}{T} \left\{ \frac{T}{2\pi T_{cz}} (l\pi)^2 + \frac{T}{2\pi T_{cy}} (k\pi)^2 + i \right\} \quad (B-25)$$

and similarly,

$$\begin{aligned}\{\bar{\epsilon}_z [\frac{(2p+1)\pi}{2h}]^2 + \bar{\epsilon}_y [\frac{(2q+1)\pi}{w}]^2 + i\sigma\} &= \frac{2\pi}{T} \left\{ \frac{T}{2\pi T_{cz}} [\frac{(2p+1)\pi}{2}]^2 + \right. \\ &\quad \left. \frac{T}{2\pi T_{cy}} [\frac{(2q+1)\pi}{2}]^2 + i \right\} \quad (B-26)\end{aligned}$$

Transforming (B-25) and (B-26) to polar form, substituting into (B-21) and putting the resulting expression for $a'_{\ell k}$ into (B-22) yields

$$c''(y,z,t) = \left(\frac{\partial \bar{c}}{\partial \xi}\right) \frac{64}{\pi^6} \frac{KT^2}{\pi^6} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \quad (B-27)$$

$$\frac{\mu_{\ell k} (-1)^{-(\ell+k)} \cos \frac{\ell \pi z}{h} \cos \frac{2k \pi y}{w} e^{i\theta_2}}{[(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2] R_1^{\frac{1}{2}}(p,q) R_2^{\frac{1}{2}}(\ell,k)}$$

which is identical to (IV-24).

B-3 Longitudinal Dispersive Mass Transport

The purpose of the following development is to present the mathematical details involved in performing the implied integrations of (IV-30) repeated here as

$$\langle \dot{m} \rangle_T = \left\langle \int_{-w/2}^{w/2} \int_{-h}^h R_e[u''(y,z,t)] \cdot R_e[c''(y,z,t)] dz dy \right\rangle_T \quad (B-28)$$

Using (IV-11) and (IV-24) for u'' and c'' respectively, the integrand of (B-28) may then be written as,

$$R_e(u'') \cdot R_e(c'') = \left(-\frac{\partial \bar{c}}{\partial \xi} \right) \cdot \frac{512 K^2 T^3}{\pi^9} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k} (-1)^{m+n-\ell-k} \cos \frac{\ell \pi z}{h} \cos \frac{2k \pi y}{w} \cos \theta_1 \cos \theta_2}{(2m+1)(2n+1)[(2p+1)^2 - (2\ell)^2][(2q+1)^2 - (2k)^2] R_1^{\frac{1}{2}}(m,n) R_1^{\frac{1}{2}}(p,q) R_2^{\frac{1}{2}}(\ell,k)} \\ \cdot \left[\cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} - g(m,n) \right] \quad (B-29)$$

where,

$$\theta_1 = \sigma t - \alpha_1(m,n)$$

$$\theta_2 = \sigma t - \alpha_1(p,q) - \alpha_2(\ell,k)$$

It is first noted that because $\mu_{00} = 0$ is required for zero contribution to c'' when both ℓ and k are zero, then the term $g(m,n)$ will not survive the double integration over y and z in (B-28) since

$$\int_{-w/2}^{w/2} \int_{-h}^h g(m,n) \cos \frac{\ell \pi z}{h} \cos \frac{2k \pi y}{w} dy dz = 0$$

if both ℓ and k are not simultaneously zero. Therefore, $g(m,n)$ will be dropped from (B-29) and will no longer be considered. The integrations over depth and width in (B-28) are then

$$\int_{-h}^h \cos \frac{\ell \pi z}{h} \cos \frac{(2m+1)\pi z}{2h} dz = \frac{4h (-1)^{m-\ell} (2m+1)}{\pi [(2m+1)^2 - (2\ell)^2]} \quad (B-30)$$

$$\int_{-w/2}^{w/2} \cos \frac{(2n+1)\pi y}{w} \cos \frac{2k \pi y}{w} dy = \frac{2w (-1)^{n-k} (2n+1)}{\pi [(2n+1)^2 - (2k)^2]} \quad (B-31)$$

using the results of the previous section. The instantaneous mass transport at any given time during the period of oscillation is obtained by substituting (B-30) and (B-31) into (B-29), or

$$\dot{m} = \left(- \frac{\partial \bar{C}}{\partial \xi} \right) \cdot \frac{4096 wh K^2 T^3}{\pi^{11}} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\cos \theta_1 \cos \theta_2 R_2^{\frac{1}{2}}(\ell, k)}{\delta_1(\ell, k, m, n, p, q)} \quad (B-32)$$

It should be noted that (B-32) could be used to obtain an expression for the dispersion coefficient as a function of time. To time average (B-32) the remaining cosine terms are expanded as follows:

$$\cos \theta_1 \cos \theta_2 = \cos (\sigma t - \alpha_1(m, n)) \cos (\sigma t - \alpha_1(p, q) - \alpha_2(\ell, k))$$

or

$$\cos\theta_1 \cos\theta_2 = \frac{1}{2} \left\{ \begin{aligned} &\cos\alpha_2(\ell, k) [\cos(2\sigma t - \alpha_1(m, n) - \alpha_1(p, q)) + \cos(\alpha_1(m, n) - \alpha_1(p, q))] \\ &+ \sin\alpha_2(\ell, k) [\sin(2\sigma t - \alpha_1(m, n) - \alpha_1(p, q)) + \sin(\alpha_1(m, n) - \alpha_1(p, q))] \end{aligned} \right\} \quad (B-33)$$

The time average over the period is then,

$$\frac{1}{T} \int_0^T \cos\theta_1 \cos\theta_2 dt = \frac{1}{2} \{ \cos\alpha_2(\ell, k) \cos \lambda + \sin\alpha_2(\ell, k) \sin \lambda \} \quad (B-34)$$

Equation (B-34) may be simplified somewhat by noting that

$$\cos\alpha_2(\ell, k) = \frac{(R_2(\ell, k) - 1)^{\frac{1}{2}}}{R_2^{\frac{1}{2}}(\ell, k)} \quad (B-35)$$

$$\sin\alpha_2(\ell, k) = \frac{1}{R_2^{\frac{1}{2}}(\ell, k)} \quad (B-36)$$

Thus, applying (B-35) and (B-36) to (B-34) and substituting the result into the time averaged form of (B-32) the expression for

$\langle \mu \rangle_T$ is

$$\langle \mu \rangle_T = \left(- \frac{\partial \bar{C}}{\partial \xi} \right) \frac{2048 \, \hbar \, K^2 T^3}{\pi^{11}} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{\mu_{\ell k} \{ (R_2(\ell, k) - 1)^{\frac{1}{2}} \cos \lambda + \sin \lambda \}}{\delta_1(\ell, k, m, n, p, q)} \quad (B-37)$$

Equation (B-37) is identical to (IV-31).

APPENDIX C

DERIVATION OF $u_{\max}$ NORMALIZING EXPRESSION FOR THREE DIMENSIONAL SHEAR FLOW SOLUTION

To obtain a relationship between the pressure gradient modulus, K , and the centerline surface amplitude of the velocity, $u_{\max}$, Equation (IV-5) is first expressed in polar form as

$$u(y,z,t) = - \frac{8KT}{\pi^3} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \quad (C-1)$$

$$\frac{(-1)^{m+n} \cos \frac{(2m+1)\pi z}{2h} \cos \frac{(2n+1)\pi y}{w} e^{i(\sigma t - \alpha_1(m,n))}}{(2m+1)(2n+1) R_1^{\frac{1}{2}}(m,n)}$$

Equation (C-1) is then evaluated at $y = z = 0$ and the real part taken,

$$R_e[u(0,0,t)] = - \frac{8KT}{\pi^3} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos(\sigma t - \alpha_1(m,n)) \quad (C-2)$$

where,

$$a_{mn} = \frac{(-1)^{m+n}}{(2m+1)(2n+1) R_1^{\frac{1}{2}}(m,n)} \quad (C-3)$$

The velocity as given by (C-2) will equal $u_{\max}$ when it is maximized with respect to time. Thus,

$$\frac{\partial}{\partial t} R_e[u(0,0,t)] = 0 = \frac{8KT\sigma}{\pi^3} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin(\sigma t - \alpha_1(m,n))$$

Expanding the composite sine term,

$$\frac{8KT\sigma}{\pi^3} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \{ \sin \sigma t \cos \alpha_1(m,n) - \cos \sigma t \sin \alpha_1(m,n) \} = 0$$

and,

$$\sin \sigma t \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos \alpha_1(m,n) = \cos \sigma t \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin \alpha_1(m,n)$$

The phase angle at which the maximum velocity occurs is then

$$\sigma t_{\max} = \tan^{-1} \left\{ \frac{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin \alpha_1(m,n)}{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos \alpha_1(m,n)} \right\} \quad (C-4)$$

Next define,

$$\phi = \left[\left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos \alpha_1(m,n) \right\}^2 + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin \alpha_1(m,n) \right\}^2 \right]^{\frac{1}{2}} \quad (C-5)$$

Then,

$$\sin \sigma t_{\max} = \frac{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin \alpha_1(m,n)}{\phi} \quad (C-6)$$

and,

$$\cos \sigma t_{\max} = \frac{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos \alpha_1(m,n)}{\phi} \quad (C-7)$$

Equation (C-2) evaluated at $t = t_{\max}$ becomes,

$$u_{\max} = -\frac{8KT}{\pi^3} \left\{ \cos \sigma t_{\max} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos \alpha_1(m,n) + \sin \sigma t_{\max} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin \alpha_1(m,n) \right\} \quad (C-8)$$

Substituting (C-6) and (C-7) into (C-8)

$$u_{\max} = -\frac{8KT}{\pi^3 \phi} \left[\left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \cos \alpha_1(m,n) \right\}^2 + \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin \alpha_1(m,n) \right\}^2 \right]$$

and,

$$u_{\max} = -\frac{8KT}{\pi^3} \left(\frac{\phi^2}{\phi} \right) \quad (C-9)$$

The desired relationship between K and $u_{\max}$ is then

$$K = -\frac{u_{\max} \pi^3}{8T\phi} \quad (C-10)$$

REFERENCES

1. Taylor, G.I., "Dispersion of Soluble Matter in Solvent Flowing Slowly Through a Tube," Proc. Roy. Soc., A, Vol. 219, pp. 186-203, 1953.
2. Taylor, G.I., "The Dispersion of Matter in Turbulent Flow Through a Pipe," Proc. Roy. Soc., A, Vol. 223, pp. 446-468, 1954.
3. Aris, R., "On the Dispersion of a Solute in a Fluid Flowing Through a Tube," Proc. Roy. Soc., A, Vol. 235, pp. 67-77, 1956.
4. Elder, J.W., "The Dispersion of Marked Fluid in Turbulent Shear Flow," J. Fluid Mech., Vol. 5, pp. 544-560, 1959.
5. Bowden, K.F., "Horizontal Mixing in the Sea Due to a Shearing Current," J. Fluid Mech., Vol. 21, pp. 83-95, 1965.
6. Fischer, H.B., "The Mechanics of Dispersion in Natural Streams," J. Hyd. Div., ASCE, Vol. 93, No. HY6, pp. 187-216, 1967.
7. Taylor, G.I., "Diffusion by Continuous Movements," Proc. Lon. Math. Soc., Ser. 2, Vol. 20, pp. 196-211, 1922.
8. Fischer, H.B., "Dispersion Predictions in Natural Streams," J. San. Eng. Div., ASCE, Vol. 94, No. SA5, pp. 927-943, 1968.
9. Holley, E.R. and D.R.F. Harleman, "Dispersion of Pollutants in Estuary Type Flows," M.I.T. Hydro. Lab. Tech. Rpt. No. 74, 1965.
10. Okubo, A., "The Effect of Shear in an Oscillatory Current on Horizontal Diffusion from an Instantaneous Source," Int. J. of Oceanology and Limnology, Vol. 1, No. 3, pp. 194-204, 1967.
11. Holley, E.R., Harleman, D.R.F. and H.B. Fischer, "Dispersion in Homogeneous Estuary Flow," J. Hyd. Div., ASCE, No. HY8, pp. 1691-1709, 1970.
12. Fischer, H.B., "Mass Transport Mechanisms in Partially Stratified Estuaries," J. Fluid Mech., Vol. 53, pp. 671-687, 1972.
13. Fischer, H.B., "Longitudinal Dispersion and Turbulent Mixing in Open-Channel Flow," in Annual Review of Fluid Mechanics, Vol. 5, pp. 59-78, 1973.

14. Awaya, Y., "Turbulent Dispersion in Periodic Flow," Proc. 13th Congress, I.A.H.R., Kyoto, Japan, Vol. 3, pp. 207-214, 1969.
15. Fischer, H.B. and E.R. Holley, "Analysis of the Use of Distorted Hydraulic Models for Dispersion Studies," Water Resour. Res., Vol. 7, No. 1, pp. 46-51, 1971.
16. Fukuoka, S., "Longitudinal Dispersion of Matter in Alternating Shear Flows," Res. Bull. No. C9, Dept. of Eng., James Cook University of North Queensland, 1973.
17. Schlichting, H., Boundary Layer Theory, 6th ed., New York: McGraw-Hill Book Co., 1968.
18. Lamb, Sir Horace, Hydrodynamics, 6th ed., New York: Dover Publications, 1945.
19. Segall, B.A., Velocity Gradients and their Influence on Dispersion in Oscillating Shear Flow, Ph.D. dissertation, New York University, at New York, June, 1971.
20. Segall, B.A. and E.R. Gidlund, "Velocity Profiles and Dispersion in Estuarine Flow," J. San. Eng. Div., ASCE, Vol. 98, No. SA4, 1972.
21. Fischer, H.B., "A Note on the One-Dimensional Dispersion Model," Int. J. Air Wat. Poll., Vol. 10, pp. 443-452, 1966.
22. Hildebrand, F.B., Advanced Calculus for Applications, Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1962.
23. Ward, P.R.B., "Transverse Dispersion in Oscillatory Channel Flow," J. Hyd. Div., ASCE, Vol. 100, No. HY6, 1974.
24. Miller, A.C. and E.V. Richardson, "Diffusion and Dispersion in Open Channel Flow," J. Hyd. Div., ASCE, Vol. 100, No. HY1, 1974.
25. Godfrey, R.G. and B.J. Frederick, "Dispersion in Natural Streams," United States Geological Survey open-file report, 1963.
26. Thackston, E.L. and P.A. Krenkel, "Longitudinal Mixing and Reaeration in Natural Streams," Tech. Rpt. No. 7, Sanitary and Water Resources Engineering, Vanderbilt University, 1966.
27. Thomas, I.E., Ph.D. dissertation, Northwestern University, Evanston, Ill., 1958.
28. Fischer, H.B., United States Geological Survey Professional Paper 582-A, 1968.
29. Yotsukura, N., Fischer, H.B., and W.W. Sayre, United States Geological Survey Water Supply Paper 1899-G, 1970.

30. Bowles, P., Burns, R.H., Hudswell, F. and R.T.P. Whipple, "Sea Disposal of Low Activity Effluent," Proc. Second Int. Conf. Peaceful Uses of Atomic Energy, U.N., New York, Vol. 18, pp. 376-389.
31. Bowden, K.F., "The Mixing Processes in a Tidal Estuary," Int. J. Air Wat. Poll., Vol. 7, pp. 343-356, 1963.
32. Fukuoka, S., "A Laboratory Study on Longitudinal Dispersion in Alternating Shear Flows," Res. Bull. No. C-12, Dept. of Eng., James Cook University of North Queensland, 1974.

BIOGRAPHICAL SKETCH

Robert Bruce Taylor III was born in 1942, in Baltimore, Maryland where he received his primary and secondary education. In 1960 he entered the United States Naval Academy from which he was graduated with distinction and commissioned as an Ensign in the United States Navy in 1964. During the period June, 1964 to June 1966 he attended various navy nuclear power training schools and the U.S. Naval Submarine School, New London, Connecticut. From July, 1966 to September, 1969 he served in USS Daniel Webster SSBN 626 (BLUE) and USS Haddo SSN 604. During this period he was qualified as a naval nuclear submarine officer. In September, 1969 he resigned his naval commission and entered the University of Miami from which he was granted the Master of Science degree in Ocean Engineering in June, 1971. From March, 1971 until August, 1974 he was employed as a graduate research assistant for the Coastal and Oceanographic Engineering Laboratory, University of Florida. During this period he pursued his Doctor of Philosophy degree in Civil Engineering which he received in August, 1974.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Robert G. Dean
Robert G. Dean, Chairman
Professor of Civil and Coastal Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Wayne C. Huber
Wayne C. Huber
Associate Professor of Environmental
Engineering Sciences

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Omar H. Shemdin
Omar H. Shemdin
Associate Professor of Civil and
Coastal Engineering

This dissertation was submitted to the Graduate Faculty of the College of Engineering and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1974

Wayne H. Chen
Dean, College of Engineering

Dean, Graduate School

